Answer on Question #61885 – Math – Linear Algebra

Question

Find the polynomial of degree 4 whose graph goes through the points (−3,−296),(−2,−60),(−2,−60), (0,4), (2,4) and (3,−110).

Solution

The general form of the polynomial of degree 4 is

$$y = f(x) = a_1x^4 + a_2x^3 + a_3x^2 + a_4x + a_5. \quad \textbf{(1)}$$

Table 1

x	y
-3	-296
-2	-260
0	4
2	4
3	-110

Substituting values from Table 1 into (1) obtain the system as follows

$$\begin{cases} a_1\cdot(-3)^4 + a_2\cdot(-3)^3 + a_3\cdot(-3)^2 + a_4\cdot(-3) + a_5 = -296, \\ a_1\cdot(-2)^4 + a_2\cdot(-2)^3 + a_3\cdot(-2)^2 + a_4\cdot(-2) + a_5 = -260, \\ a_1\cdot(0)^4 + a_2\cdot(0)^3 + a_3\cdot(0)^2 + a_4\cdot(0) + a_5 = 4, \\ a_1\cdot(2)^4 + a_2\cdot(2)^3 + a_3\cdot(2)^2 + a_4\cdot(2) + a_5 = 4, \\ a_1\cdot(3)^4 + a_2\cdot(3)^3 + a_3\cdot(3)^2 + a_4\cdot(3) + a_5 = -110, \end{cases}$$

$$\begin{cases} 81a_1 - 27a_2 + 9a_3 - 3a_4 + a_5 = -296, \\ 16a_1 - 8a_2 + 4a_3 - 2a_4 + a_5 = -260, \\ 0\cdot a_1 + 0\cdot a_2 + 0\cdot a_3 + 0\cdot a_4 + a_5 = 4, \\ 16a_1 + 8a_2 + 4a_3 + 2a_4 + a_5 = 4, \\ 81a_1 + 27a_2 + 9a_3 + 3a_4 + a_5 = -110, \end{cases}$$

$$\begin{pmatrix} 81 & -27 & 9 & -3 & 1 \\ 16 & -8 & 4 & -2 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 16 & 8 & 4 & 2 & 1 \\ 81 & 27 & 9 & 3 & 1 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{pmatrix} = \begin{pmatrix} -296 \\ -260 \\ 4 \\ 4 \\ -110 \end{pmatrix} \quad \textbf{(2)}$$

Let $C = \begin{pmatrix} 81 & -27 & 9 & -3 & 1 \\ 16 & -8 & 4 & -2 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 16 & 8 & 4 & 2 & 1 \\ 81 & 27 & 9 & 3 & 1 \end{pmatrix}$.

Then the following matrix can be computed by means of the function MINVERSE from Microsoft Excel:

$$C^{-1} = \frac{1}{360}\begin{pmatrix} 4 & -9 & 10 & -9 & 4 \\ -12 & 18 & 0 & -18 & 12 \\ -16 & 81 & -130 & 81 & -16 \\ 48 & -162 & 0 & 162 & -48 \\ 0 & 0 & 360 & 0 & 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{90} & -\frac{1}{40} & \frac{1}{36} & -\frac{1}{40} & \frac{1}{90} \\ -\frac{1}{30} & \frac{1}{20} & 0 & -\frac{1}{20} & \frac{1}{30} \\ -\frac{2}{45} & \frac{9}{40} & -\frac{13}{36} & \frac{9}{40} & -\frac{2}{45} \\ \frac{2}{15} & -\frac{9}{20} & 0 & \frac{9}{20} & -\frac{2}{15} \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix} \approx$$

$$\approx \begin{pmatrix} 0.011 & -0.025 & 0.028 & -0.025 & 0.011 \\ -0.033 & 0.05 & 0 & -0.05 & 0.033 \\ -0.044 & 0.225 & -0.361 & 0.225 & -0.044 \\ 0.133 & -0.45 & 0 & 0.45 & -0.133 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}.$$

Identities

$$C^{-1}C = E = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \textbf{(3)}$$

$$E\begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{pmatrix} \textbf{(4)}$$

hold true.

Multiplying (2) on the left by C^{-1} and using (3), (4) and function MMULT from Microsoft Excel obtain

$$\begin{pmatrix} \frac{1}{90} & -\frac{1}{40} & \frac{1}{36} & -\frac{1}{40} & \frac{1}{90} \\ -\frac{1}{30} & \frac{1}{20} & 0 & -\frac{1}{20} & \frac{1}{30} \\ -\frac{2}{45} & \frac{9}{40} & -\frac{13}{36} & \frac{9}{40} & -\frac{2}{45} \\ \frac{2}{15} & -\frac{9}{20} & 0 & \frac{9}{20} & -\frac{2}{15} \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}\begin{pmatrix} 81 & -27 & 9 & -3 & 1 \\ 16 & -8 & 4 & -2 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 16 & 8 & 4 & 2 & 1 \\ 81 & 27 & 9 & 3 & 1 \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{pmatrix} =$$

$$= \begin{pmatrix} \frac{1}{90} & -\frac{1}{40} & \frac{1}{36} & -\frac{1}{40} & \frac{1}{90} \\ -\frac{1}{30} & \frac{1}{20} & 0 & -\frac{1}{20} & \frac{1}{30} \\ -\frac{2}{45} & \frac{9}{40} & -\frac{13}{36} & \frac{9}{40} & -\frac{2}{45} \\ \frac{2}{15} & -\frac{9}{20} & 0 & \frac{9}{20} & -\frac{2}{15} \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}\begin{pmatrix} -296 \\ -260 \\ 4 \\ 4 \\ -110 \end{pmatrix},$$

$$\begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{pmatrix} = \begin{pmatrix} 2 \\ -7 \\ -41 \\ 94 \\ 4 \end{pmatrix}.$$

Thus, the polynomial of degree 4 whose graph goes through the points (−3,−296), (−2,−60), (−2,−60), (0,4), (2,4) and (3,−110) will be $2x^4 - 7x^3 - 41x^2 + 94x + 4$.

Answer: $2x^4 - 7x^3 - 41x^2 + 94x + 4$.