Answer on Question #61283 –Math– Calculus

Question

Derive the equation of the right circular cone generated by rotating the line 6x=3y=2z about the line 2x=-2y=-z.

Solution

Write the equation of a line $2x = -2y = -z$ as $\frac{x}{\frac{1}{2}} = \frac{y}{-\frac{1}{2}} = \frac{z}{-1}$. The equation of plane perpendicular to given line has the form $x - y - 2z + D = 0$, where D – constant.

Line represented by equation $\frac{x-x_0}{m} = \frac{y-y_0}{p} = \frac{z-z_0}{q}$ ($\vec{s}(m,p,q)$ is the direction vector of the straight line) and plane $Ax + By + Cz + D = 0$ ($\vec{n}(A,B,C)$ is a normal vector of plane). Hence we get the condition for line and plane $Am + Bp + Cq = 0$ to be parallel. Line is perpendicular to the plane only if the direction vector of the straight collinear to the normal vector of this plane:

$\frac{A}{m} = \frac{B}{p} = \frac{C}{q}$.

Plane $x - y - 2z + D = 0$ has normal vector $(1,-1,-2)$, and line $2x = -2y = -z$ or $\frac{x}{\frac{1}{2}} = \frac{y}{-\frac{1}{2}} = \frac{z}{-1}$ has direction vector $\left(\frac{1}{2}, -\frac{1}{2}, -1\right)$. Vectors are collinear, this means that the plane and the straight line are perpendicular.

Find the point of intersection of lines $2x = -2y = -z$ and $6x = 3y = 2z$.

$\begin{cases} 6x = 3y = 2z \\ 2x = -2y = -z \end{cases} \rightarrow (0,0,0)$ – the center of the cone.

Choose some point on the line $2x = -2y = -z$. Let $z = h$, hence $y = \frac{h}{2}$, $x = -\frac{h}{2}$. Find the equation of the plane perpendicular to the axis of rotation passing through the point.

$-\frac{h}{2} - \frac{h}{2}y - 2h + D = 0 \rightarrow D = 3h \rightarrow x - y - 2z + 3h = 0$

Find the point of intersection of line $6x = 3y = 2z$ and plane $x - y - 2z + 3h = 0$.

$\begin{cases} 6x = 3y = 2z \\ x - y - 2z + 3h = 0 \end{cases}$

Let $z = 6t$, where t some number. From first equation get $t = 4y$, $x = 2t$. Substituting the obtained values into the second equation of the system get

$2t - 4t - 12t + 3h = 0 \rightarrow 3h = 14t\ t = \frac{3h}{14}$. Therefore $x = \frac{3h}{14}, y = \frac{6h}{7}, z = \frac{9h}{7}$.

Find the radius of rotation for that point around the axis $2x = -2y = -z$ using distance formula between two point $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$.

$R = \sqrt{\left(\frac{3h}{7} + \frac{h}{2}\right)^2 + \left(\frac{6h}{7} - \frac{h}{2}\right)^2 + \left(\frac{9h}{7} - h\right)^2} = \sqrt{\frac{210}{196}}h.$

Write the equation of the circle of rotation for the point $x = \frac{3h}{14}, y = \frac{6h}{7}, z = \frac{9h}{7}$.

$$\begin{cases} x - y - 2z + 3h = 0 \\ \left(x + \frac{h}{2}\right)^2 + \left(y - \frac{h}{2}\right)^2 + (z - h)^2 = \frac{210}{196} h^2 \end{cases}$$

We find from the first equation h, and substitute into the second equation and get:

$h = \frac{y+2z-x}{3}$

$$\left(x + \frac{y + 2z - x}{6}\right)^2 + \left(y - \frac{y + 2z - x}{6}\right)^2 + \left(z - \frac{y + 2z - x}{3}\right)^2 = \frac{210}{196}\left(\frac{y + 2z - x}{3}\right)^2$$

$$\left(\frac{y + 2z + 5x}{6}\right)^2 + \left(\frac{5y - 2z + x}{6}\right)^2 + \left(\frac{z - y + x}{3}\right)^2 = \frac{5(y + 2z - x)^2}{42}$$

As result get

$$7(y + 2z + 5x)^2 + 7(5y - 2z + x)^2 + 28(z - y + x)^2 = 30(y + 2z - x)^2$$

$$7(y^2 + 4z^2 + 25x^2 + 4yz + 10xy + 20zx) + 7(25y^2 + 4z^2 + x^2 - 20yz + 10xy - 4xz) +$$

$$+28(z^2 + y^2 + x^2 - 2zy - 2xy + 2xz) = 30(4z^2 + y^2 + x^2 + 4zy - 2xy - 4xz)$$

$$180y^2 + 180x^2 - 36z^2 - 288zy + 144xy + 288xz = 0$$

Thus, equation of cone is

$$5y^2 + 5x^2 - z^2 - 8zy + 4xy + 8xz = 0$$

Answer: $5y^2 + 5x^2 - z^2 - 8zy + 4xy + 8xz = 0$.