Answer on Question #60923 – Math – Calculus

Question

Use the formula $\cos\theta = \frac{1}{2}(e^{i\theta} + e^{-i\theta})$ to obtain the identity

$$\cos^5\theta = \frac{1}{16}(\cos 5\theta + 5\cos 3\theta + 10\cos\theta).$$

Solution

According to the binomial expansion,

$(a+b)^n = \sum_{k=0}^{n} C_n^k a^{n-k} b^k$, where $C_n^k = \frac{n!}{(n-k)!k!}$.

$$(a+b)^5 = \sum_{k=0}^{5} C_5^k a^{5-k} b^k = C_5^0 a^5 b^0 + C_5^1 a^4 b^1 + C_5^2 a^3 b^2 + C_5^3 a^2 b^3 + C_5^4 a^1 b^4 + C_5^5 a^0 b^5 =$$

$$= a^5 + 5a^4 b + 10a^3 b^2 + 10a^2 b^3 + 5ab^4 + b^5.$$

Then, let's apply the binomial expansion,

$$\cos^5\theta = \left(\frac{1}{2}(e^{i\theta} + e^{-i\theta})\right)^5 = \frac{1}{2^5}(e^{i\theta} + e^{-i\theta})^5 =$$

$$= \frac{1}{32}(e^{5i\theta} + 5e^{4i\theta}e^{-i\theta} + 10e^{3i\theta}e^{-2i\theta} + 10e^{2i\theta}e^{-3i\theta} + 5e^{i\theta}e^{-4i\theta} + e^{-5i\theta}) =$$

$$= \frac{1}{32}(e^{5i\theta} + 5e^{3i\theta} + 10e^{i\theta} + 10e^{-i\theta} + 5e^{-3i\theta} + e^{-5i\theta}) =$$

$$= \frac{1}{16}\left(\frac{1}{2}(e^{5i\theta} + e^{-5i\theta}) + \frac{1}{2}(5e^{3i\theta} + 5e^{-3i\theta}) + \frac{1}{2}(10e^{i\theta} + 10e^{-i\theta})\right) =$$

$$= \frac{1}{16}\Big(\underbrace{\frac{1}{2}(e^{5i\theta} + e^{-5i\theta})}_{\cos 5\theta} + 5\cdot\underbrace{\frac{1}{2}(e^{3i\theta} + e^{-3i\theta})}_{\cos 3\theta} + 10\cdot\underbrace{\frac{1}{2}(e^{i\theta} + 10e^{-i\theta})}_{\cos\theta}\Big) =$$

$$= \frac{1}{16}(\cos 5\theta + 5\cos 3\theta + 10\cos\theta).$$