Answer on Question #60894 – Math – Calculus

Question

a) use the binomial series to find the Taylor series about 0 for the function f(x)=(1+x)^-1/5, giving all terms up to the one in x^4, and calculating each coefficient as an integer or a fraction.

b) state an interval of validity for this series.

c) use this series and the Taylor series for cosx to find the cubic Taylor polynomial about 0 for the function f(x)= cosx/(1+x)^1/5.

Solution

a) The Taylor series about 0 for the function $(1+x)^{\alpha}$ is

$(1+x)^{\alpha} = \sum_{n=0}^{\infty} \binom{\alpha}{n} x^n$,

where $\binom{\alpha}{n}$ is generalized binomial coefficients:

$\binom{\alpha}{n} = \prod_{k=1}^{n} \frac{\alpha-k+1}{k} = \frac{\alpha(\alpha-1)(\alpha-2)\cdots(\alpha-n+1)}{n!}$.

Hence $\binom{\alpha}{1} = \alpha$, $\binom{\alpha}{2} = \frac{\alpha(\alpha-1)}{2!} = \frac{\alpha(\alpha-1)}{2}$, $\binom{\alpha}{3} = \frac{\alpha(\alpha-1)(\alpha-2)}{3!} = \frac{\alpha(\alpha-1)(\alpha-2)}{6}$,

$\binom{\alpha}{4} = \frac{\alpha(\alpha-1)(\alpha-2)(\alpha-3)}{4!} = \frac{\alpha(\alpha-1)(\alpha-2)(\alpha-3)}{24}$.

For $\alpha = -\frac{1}{5}$ we shall have

$$(1+x)^{-\frac{1}{5}} = 1 - \frac{1}{5}x + \frac{3}{25}x^2 - \frac{11}{125}x^3 + \frac{44}{625}x^4 - \cdots$$

b) Interval of validity for this series is $|x| < 1$.

c) The Taylor series about 0 for the function cos(x) is

$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \cdots$

and for function $f(x) = \frac{\cos(x)}{(1+x)^{\frac{1}{5}}}$ we have

$$f(x) = \frac{\cos(x)}{(1+x)^{\frac{1}{5}}} = \cos(x)\,(1+x)^{-\frac{1}{5}} = \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots\right)\left(1 - \frac{1}{5}x + \frac{3}{25}x^2 - \frac{11}{125}x^3 + \frac{44}{625}x^4 - \cdots\right)$$

$$= 1 - \frac{1}{5}x + \frac{3}{25}x^2 - \frac{11}{125}x^3 + \frac{44}{625}x^4 - \frac{x^2}{2!} + \frac{x^3}{5\cdot 2!} - \frac{3x^4}{25\cdot 2!} + \frac{11x^5}{125\cdot 2} - \frac{44}{625\cdot 2}x^6 + \frac{x^4}{4!} - \frac{1}{5\cdot 4!}x^5 + \cdots$$

$$= 1 - \frac{1}{5}x + \left(\frac{3}{25} - \frac{1}{2}\right)x^2 + \left(-\frac{11}{125} + \frac{1}{5\cdot 2!}\right)x^3 + \left(\frac{44}{625} - \frac{3}{25\cdot 2!} + \frac{1}{4!}\right)x^4 + \cdots$$

$$= 1 - \frac{1}{5}x - \frac{19}{50}x^2 + \frac{3}{250}x^3 + \frac{781}{15000}x^4 + \cdots$$

Answer: a) $1 - \frac{1}{5}x + \frac{3}{25}x^2 - \frac{11}{125}x^3 + \frac{44}{625}x^4 - \cdots$; **b)** $|x| < 1$; **c)** $1 - \frac{1}{5}x - \frac{19}{50}x^2 + \frac{3}{250}x^3 + \frac{781}{15000}x^4 + \cdots$.