Answer on Question #60834 – Math – Linear Algebra

Question

a) The Gauss elimination method is used to solve the system of equations
6 x 4x x 1 + 2 + α 3 =
3 2x x 2 x 1 − 2 + α 3 =
5 x 3x x α 1 + 2 + 3 =
Find the value of α for which the system has

(i) a unique solution;

(ii) no solution;

(iii) infinitely many solutions.

Solution

We have the system of equation

$$\begin{cases} x_1 + 4x_2 + \alpha x_3 = 6, \\ 2x_1 - x_2 + 2\alpha x_3 = 3, \\ \alpha x_1 + 3x_2 + x_3 = 5. \end{cases}$$

or in the form $Ax = b$

$$\begin{pmatrix} 1 & 4 & \alpha \\ 2 & -1 & 2\alpha \\ \alpha & 3 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 6 \\ 3 \\ 5 \end{pmatrix}.$$

We shall use the Gauss elimination method to solve the system. So put the augmented matrix into the upper triangular form. Firstly, subtract the first row multiplied by 2 from the second row and divide the second row by -9:

$$\left(\begin{array}{ccc|c} 1 & 4 & \alpha & 6 \\ 2 & -1 & 2\alpha & 3 \\ \alpha & 3 & 1 & 5 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 4 & \alpha & 6 \\ 0 & -9 & 0 & -9 \\ \alpha & 3 & 1 & 5 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 4 & \alpha & 6 \\ 0 & 1 & 0 & 1 \\ \alpha & 3 & 1 & 5 \end{array}\right).$$

Then subtract the first row multiplied by α from the third row and then divide the third row by (-1):

$$\left(\begin{array}{ccc|c} 1 & 4 & \alpha & 6 \\ 0 & 1 & 0 & 1 \\ \alpha & 3 & 1 & 5 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 4 & \alpha & 6 \\ 0 & 1 & 0 & 1 \\ 0 & 3-4\alpha & 1-\alpha^2 & 5-6\alpha \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 4 & \alpha & 6 \\ 0 & 1 & 0 & 1 \\ 0 & 4\alpha-3 & \alpha^2-1 & 6\alpha-5 \end{array}\right).$$

In the next step we subtract the second row multiplied by $(4\alpha - 3)$ from the third row:

$$\left(\begin{array}{ccc|c} 1 & 4 & \alpha & 6 \\ 0 & 1 & 0 & 1 \\ 0 & 4\alpha-3 & \alpha^2-1 & 6\alpha-5 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 4 & \alpha & 6 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & \alpha^2-1 & 2\alpha-2 \end{array}\right).$$

Then subtract the second row multiplied by 4 from the first row:

$$\begin{pmatrix} 1 & 4 & \alpha & | & 6 \\ 0 & 1 & 0 & | & 1 \\ 0 & 0 & \alpha^2 - 1 & | & 2\alpha - 2 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & \alpha & | & 2 \\ 0 & 1 & 0 & | & 1 \\ 0 & 0 & \alpha^2 - 1 & | & 2\alpha - 2 \end{pmatrix}.$$

Now we need to find the value of α for which system has **(i)** unique solution.

From the third row it follows that

$$x_3 = \frac{2\alpha-2}{\alpha^2-1} = \frac{2}{\alpha+1}, \quad \alpha \neq \pm 1.$$

From the second row it follows that

$$x_2 = 1.$$

From the first row it follows that

$$x_1 + \frac{2\alpha}{\alpha+1} = 2;$$

$$x_1 = 2 - \frac{2\alpha}{\alpha+1} = \frac{2\alpha+2-2\alpha}{\alpha+1} = \frac{2}{\alpha+1}.$$

So if $\alpha \neq \pm 1$, we have the unique solution.

(iii) If $\alpha = 1$, then the system will be

$$\begin{pmatrix} 1 & 0 & 1 & | & 2 \\ 0 & 1 & 0 & | & 1 \\ 0 & 0 & 0 & | & 0 \end{pmatrix}.$$

In other words, we have only two equations for three variables. In this case the system has infinitely many solutions.

(ii) If $\alpha = -1$, then the system has no solution, because the third row yields $0 = -4$, which is false.

Answer: (i) $\alpha \in \mathbb{R} \setminus \{-1,1\}$, **(ii)** $\alpha = -1$, **(iii)** $\alpha = 1$.