Answer on Question #60791 – Math – Calculus

Question

The movement of the crest of a wave is modelled with the equation h(t)=0.3cos3t+0.4sin3t. Find the maximum height of the wave and the time at which it occurs.

Solution

Method 1

We have a function

h(t) = 0.3 cos3t + 0.4 sin3t

1)Find the derivative of the function:

h'(t) = -0.9 sin 3t + 1.2 cos3t

2)Determine the critical points:

h'(t) = 0:

-0.9 sin3t + 1.2 cos3t = 0

0.9 sin3t = 1.2 cos3t

cos3t $\neq$ 0:

$0.9\frac{\sin 3t}{\cos 3t} = 1.2$

tan 3t = $\frac{4}{3}$

3t = arctan ($\frac{4}{3}$)$+n\pi$, where n is integer.

The critical points are

$t=\frac{1}{3}\left(\arctan\left(\frac{4}{3}\right)+n\pi\right)$.

If $n=0$, then $t_1=\frac{1}{3}\left(\arctan\left(\frac{4}{3}\right)\right)\approx 0.31$ lies in the first quadrant;

$3t_1=\left(\arctan\left(\frac{4}{3}\right)\right)\approx 0.93$ lies in the first quadrant;

$$\cos(3t_1)=\frac{1}{\sqrt{1+tan^2(3t_1)}}=\frac{1}{\sqrt{1+\left(\frac{4}{3}\right)^2}}=\frac{3}{5};$$

$$sin(3t_1)=\frac{\tan(3t_1)}{\sqrt{1+tan^2(3t_1)}}=\frac{\frac{4}{3}}{\sqrt{1+\left(\frac{4}{3}\right)^2}}=\frac{4}{5}.$$

If $n = 1$, then $t_2 = \frac{1}{3}\left(\arctan\left(\frac{4}{3}\right) + \pi\right) \approx 1.36$ lies in the first quadrant;

$3t_2 = \left(\arctan\left(\frac{4}{3}\right) + \pi\right) \approx 4.07$ lies in the third quadrant;

$$\cos(3t_2) = -\frac{1}{\sqrt{1+tan^2(3t_2)}} = -\frac{1}{\sqrt{1+\left(\frac{4}{3}\right)^2}} = -\frac{3}{5};$$

$$sin(3t_2) = -\frac{\tan(3t_2)}{\sqrt{1+tan^2(3t_2)}} = -\frac{\frac{4}{3}}{\sqrt{1+\left(\frac{4}{3}\right)^2}} = -\frac{4}{5}.$$

If $n = 2$, then $t_3 = \frac{1}{3}\left(\arctan\left(\frac{4}{3}\right) + 2\pi\right) \approx 2.4$ lies in the second quadrant;

$3t_3 = \left(\arctan\left(\frac{4}{3}\right) + 2\pi\right) \approx 7.21$ lies in the first quadrant;

$$\cos(3t_3) = \frac{1}{\sqrt{1+tan^2(3t_3)}} = \frac{1}{\sqrt{1+\left(\frac{4}{3}\right)^2}} = \frac{3}{5};$$

$$sin(3t_3) = \frac{\tan(3t_2)}{\sqrt{1+tan^2(3t_2)}} = \frac{\frac{4}{3}}{\sqrt{1+\left(\frac{4}{3}\right)^2}} = \frac{4}{5}.$$

If $n = 3$, then $t_4 = \frac{1}{3}\left(\arctan\left(\frac{4}{3}\right) + 3\pi\right) \approx 3.45$ lies in the third quadrant;

$3t_4 = \left(\arctan\left(\frac{4}{3}\right) + 3\pi\right) \approx 10.35$ lies in the third quadrant;

$$\cos(3t_4) = -\frac{1}{\sqrt{1+tan^2(3t_4)}} = -\frac{1}{\sqrt{1+\left(\frac{4}{3}\right)^2}} = -\frac{3}{5};$$

$$sin(3t_3) = -\frac{\tan(3t_2)}{\sqrt{1+tan^2(3t_2)}} = -\frac{\frac{4}{3}}{\sqrt{1+\left(\frac{4}{3}\right)^2}} = -\frac{4}{5}.$$

If $n = 4$, then $t_5 = \frac{1}{3}\left(\arctan\left(\frac{4}{3}\right) + 4\pi\right) \approx 5.00$ lies in the fourth quadrant;

$3t_5 = \left(\arctan\left(\frac{4}{3}\right) + 4\pi\right) \approx 13.49$ lies in the first quadrant;

$$\cos(3t_5) = \frac{1}{\sqrt{1+tan^2(3t_4)}} = \frac{1}{\sqrt{1+\left(\frac{4}{3}\right)^2}} = \frac{3}{5}.$$

$sin(3t_3) = \frac{\tan(3t_2)}{\sqrt{1+tan^2(3t_2)}} = \frac{\frac{4}{3}}{\sqrt{1+\left(\frac{4}{3}\right)^2}} = \frac{4}{5}$ and so on.

In any case we obtain either

i) $\cos(3t^*) = \frac{3}{5}$, $\sin(3t^*) = \frac{4}{5}$, when n is even

or

ii) $\cos(3t_*) = -\frac{3}{5}$, $\sin(3t_*) = -\frac{4}{5}$, when n is odd.

3) Find the second derivative

h''(t) = -2.7 cos 3t - 3.6 sin3t

h''(t^*) =− 2.7 cos 3t^* −3.6 sin3t^* =−2.7 $\cdot \frac{3}{5}$ −3.6$\cdot \frac{4}{5} < 0$, hence

$t^* = \frac{1}{3}\left(\arctan\left(\frac{4}{3}\right) + 2n\pi\right)$ is maximum, where n is integer, then

h(t^*) = 0.3 cos3t^*+ 0.4 sin3t^*=0.3 $\cdot \frac{3}{5} + 0.4 \cdot \frac{4}{5} = \frac{0.9+1.6}{5} = 0.5$,

h''(t_*) =− 2.7 cos 3t_* −3.6 sin3t_* =−2.7 $\cdot \left(-\frac{3}{5}\right)$ −3.6$\cdot \left(-\frac{4}{5}\right) > 0$, hence

$t^* = \frac{1}{3}\left(\arctan\left(\frac{4}{3}\right) + (2n+1)\pi\right)$ is minimum, where n is integer, then

h(t_*) = 0.3 cos3t_*+ 0.4 sin3t_*=0.3 $\cdot \left(-\frac{3}{5}\right) + 0.4 \cdot \left(-\frac{4}{5}\right) = \frac{-0.9-1.6}{5} = -0.5$.

The maximum height of the wave equals 0.5 and the time at which it occurs is $\frac{1}{3}\left(\arctan\left(\frac{4}{3}\right) + 2n\pi\right)$, $n \geq 0$, n is integer.

Answer: 0.5; $\frac{1}{3}\left(\arctan\left(\frac{4}{3}\right) + 2n\pi\right)$.

Method 2

Let

h(t) = 0.3 cos $(3t)$ + 0.4 sin$(3t)$=$R\sin(3t+\alpha) = R\sin(3t)\cdot\cos(\alpha) + R\cos(3t)\cdot\sin(\alpha)$,

hence

for $\cos(3t)$ obtain the identity

$R \cdot \sin(\alpha) = 0.3$ (1)

for $\sin(3t)$ obtain the identity

$R \cdot \cos(\alpha) = 0.4$ (2)

It follows from (1) and (2) that $\tan(\alpha) = \frac{R\cdot\sin(\alpha)}{R\cdot\cos(\alpha)} = \frac{3}{4}$.

So $\alpha = \arctan\frac{3}{4}$

It follows from (1) and (2) that $R = \sqrt{R^2 sin^2(\alpha) + R^2 cos^2(\alpha)} = \sqrt{0.3^2 + 0.4^2} = 0.5$.

Besides, $|\sin(3t + \alpha)| \leq 1$ and the maximum height of the wave equals $R = 0.5$.

The time at which the maximum occurs is given by

$$3t + \alpha = \frac{\pi}{2} + 2k\pi, k \text{ is integer},$$

$$t = \frac{1}{3}\left(\frac{\pi}{2} + 2k\pi - \alpha\right) = \frac{1}{3}\left(\frac{\pi}{2} + 2k\pi - \arctan\frac{3}{4}\right) = \frac{1}{3}\left(\left(\frac{\pi}{2} - \arctan\frac{3}{4}\right) + 2k\pi\right) =$$

$$= \frac{1}{3}\left(\operatorname{arccot}\frac{3}{4} + 2k\pi\right) = \frac{1}{3}\left(\arctan\frac{4}{3} + 2k\pi\right).$$

The maximum height of the wave equals 0.5 and the time at which it occurs is

$\frac{1}{3}\left(\arctan\frac{4}{3} + 2k\pi\right)$, $k \geq 0$, k is integer.

Answer: 0.5; $\frac{1}{3}\left(\arctan\left(\frac{4}{3}\right) + 2n\pi\right)$.