Answer on Question #60757 – Math – Differential Equations

Question

1. Solve $\frac{dy}{dx} = y^3 + 3\frac{x^2y}{x^3} + 3xy^2$

Solution

$\frac{dy}{dx} = y^3 + \frac{x^2y}{x^3} + 3xy^2$ is a linear differential equation; $\frac{dy}{dx} = y^3 + \frac{y}{x} + 3xy^2$.

Let $u = \frac{y}{x}$, then $y = ux, \frac{dy}{dx} = y' = u'x + u$.

$u'x + u = u^3x^3 + u + 3xu^2x^2;$

$u'x = u^3x^3 + 3u^2x^3;$

$u' = x^2(u^3 + 3u^2)$

$\frac{du}{dx} = x^2(u^3 + 3u^2);$

Separate the variables:

$\frac{du}{u^3+3u^2} = x^2dx;$

Integrate both sides:

$\int \frac{du}{u^3+3u^2} = \int x^2dx;$

$\frac{1}{u^3+3u^2} = \frac{1}{u^2(u+3)} = \frac{A}{u} + \frac{B}{u^2} + \frac{C}{u+3} = \frac{Au(u+3)+B(u+3)+Cu^2}{u^3+3u^2} = \frac{(A+C)u^2+(3A+B)u+3B}{u^3+3u^2};$

Solve the system $\begin{cases} A + C = 0, \\ 3A + B = 0, \\ 3B = 1, \end{cases} \Longrightarrow \begin{cases} B = \frac{1}{3}, \\ A = -\frac{B}{3} = -\frac{1}{9}, \\ C = -A = \frac{1}{9}. \end{cases}$

$$\int \frac{du}{u^3 + 3u^2} = \int \left(-\frac{1}{9u} + \frac{1}{3u^2} + \frac{1}{9(u+3)}\right) du = -\frac{1}{9}\int \frac{du}{u} + \frac{1}{3}\int \frac{du}{u^2} + \frac{1}{9}\int \frac{du}{u+3} =$$

$$= -\frac{1}{9}\ln|u| - \frac{1}{3u} + \frac{1}{9}\ln|u+3| + C = \frac{1}{9}\ln\left|\frac{u+3}{u}\right| - \frac{1}{3u} + C = \frac{1}{9}\ln\left|1 + \frac{3}{u}\right| - \frac{1}{3u} + C.$$

Then

$\frac{1}{9}\ln\left|1 + \frac{3x}{y}\right| - \frac{x}{3y} = \frac{x^3}{3} + C \Longrightarrow \frac{1}{3}\ln\left|1 + \frac{3x}{y}\right| - \frac{x}{y} - x^3 = C.$

Answer: $\frac{1}{3}\ln\left|1 + \frac{3x}{y}\right| - \frac{x}{y} - x^3 = C.$