Answer on Question #60706 - Math - Differential Equations

Question

Solve:

$$2\frac{dy}{dx} + y \cdot \tan x = \frac{(4x+5)^2}{\cos x} y^3.$$

Solution

This is the Bernoulli equation:

$$2y' + y\tan x = \frac{(4x+5)^2}{\cos x} y^3 . (\mathbf{1})$$

We first notice that $y = 0$ is a solution. Division by y^3 yields:

$$\frac{2y'}{y^3} + \frac{\tan x}{y^2} = \frac{(4x+5)^2}{\cos x} . (\mathbf{2})$$

Changing variables gives the equations:

$$z = \frac{1}{y^2};\ (\mathbf{3})\ and\ z' = -\frac{2y'}{y^3}; (\mathbf{4})$$

$$-z' + z\tan x = \frac{(4x+5)^2}{\cos x} . (\mathbf{5})$$

We use the method of variation of parameters:

$$-z' + z\tan x = 0; (\mathbf{6})$$

$$\frac{dz}{z} = \tan x\ dx. (\mathbf{7})$$

Because: $\tan x dx = \frac{\sin x\, dx}{\cos x} = -\frac{d(\cos x)}{\cos x}, (\mathbf{8})$

$$\frac{dz}{z} = -\frac{d(\cos x)}{\cos x};\ (\mathbf{9})$$

$$\ln z = -\ln(\cos x) + \ln C; (\mathbf{10})$$

$$z = \frac{C(x)}{\cos x} . (\mathbf{11})$$

Differentiation yields: $z' = \frac{C'(x)}{\cos x} + \frac{C(x)\sin x}{cos^2 x} . (\mathbf{12})$

Substituting (11) and (12) back into equation (5):

$$-\frac{C'(x)}{\cos x} - \frac{C(x)\sin x}{cos^2 x} + \frac{C(x)\sin x}{cos^2 x} = \frac{(4x+5)^2}{\cos x}; (\mathbf{13})$$

$$-\frac{C'(x)}{\cos x}=\frac{(4x+5)^2}{\cos x}.(\mathbf{14})$$

$$C'(x)=-(4x+5)^2.(\mathbf{15})$$

Integrate (15) to find C(x):

$C(x)=-\int(4x+5)^2dx=-\frac{1}{4}\int(4x+5)^2d(4x)=-\frac{1}{4}\int(4x+5)^2d(4x+5)=-\frac{1}{4\cdot3}\int 3(4x+5)^2d(4x+5)=-\frac{1}{12}(4x+5)^3+\tilde{C}=-\frac{1}{12}((4x+5)^3+C)$, $C=const.$

$$C(x)=-\frac{1}{12}((4x+5)^3+C),C=const.\ (\mathbf{16})$$

Substituting (16) back into equation (11):

$$z=-\frac{1}{12}\cdot\frac{(4x+5)^3+C}{\cos x};(\mathbf{17})$$

Substituting (3) back into equation (17):

$$\frac{1}{y^2}=-\frac{1}{12}\cdot\frac{(4x+5)^3+C}{\cos x};(\mathbf{18})$$

$$y^2=-\frac{12\cos x}{(4x+5)^3+C}=\frac{12\cos x}{C_1-(4x+5)^3};(\mathbf{19})$$

$$y=\pm2\sqrt{\frac{3\cos x}{C_1-(4x+5)^3}}.(\mathbf{20})$$

Answer:

$$y=\pm2\sqrt{\frac{3\cos x}{C_1-(4x+5)^3}};y=0.$$