Answer on Question #60687 – Math – Linear Algebra

Question

4. a) The Gauss elimination method is used to solve the system of equations

$$\begin{aligned} x_1 + 4x_2 + \alpha x_3 &= 6 \\ 2x_1 - x_2 + 2\alpha x_3 &= 3 \\ \alpha x_1 + 3x_2 + x_3 &= 5 \end{aligned} \qquad (1)$$

Find the value of α for which the system has

(i) a unique solution
(ii) no solution
(iii) infinitely many solutions.

Solution

Using the Gauss elimination method

$$\left(\begin{array}{ccc|c} 1 & 4 & \alpha & 6 \\ 2 & -1 & 2\alpha & 3 \\ \alpha & 3 & 1 & 5 \end{array}\right) \to |R_2 \leftarrow R_2 - 2R_1| \to \left(\begin{array}{ccc|c} 1 & 4 & \alpha & 6 \\ 0 & -9 & 0 & -9 \\ \alpha & 3 & 1 & 5 \end{array}\right) \to |R_2 \leftarrow R_2/(-9)| \to$$

$$\to \left(\begin{array}{ccc|c} 1 & 4 & \alpha & 6 \\ 0 & 1 & 0 & 1 \\ \alpha & 3 & 1 & 5 \end{array}\right) \to |R_3 \leftarrow R_3 - \alpha R_1| \to \left(\begin{array}{ccc|c} 1 & 4 & \alpha & 6 \\ 0 & 1 & 0 & 1 \\ 0 & 3-4\alpha & 1-\alpha^2 & 5-6\alpha \end{array}\right) \to$$

$$\to |R_3 \leftarrow R_3 - (3-4\alpha)R_2| \to \left(\begin{array}{ccc|c} 1 & 4 & \alpha & 6 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1-\alpha^2 & 5-6\alpha-(3-4\alpha) \end{array}\right) \to$$

$$\to \left(\begin{array}{ccc|c} 1 & 4 & \alpha & 6 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1-\alpha^2 & 2-2\alpha \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 4 & \alpha & 6 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & (1-\alpha)(1+\alpha) & 2(1-\alpha) \end{array}\right).$$

<u>**Case 1**</u>. Let $\alpha = 1$. Then

$$\left(\begin{array}{ccc|c} 1 & 4 & \alpha & 6 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & (1-\alpha)(1+\alpha) & 2(1-\alpha) \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 4 & 1 & 6 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right) \to |R_1 \leftarrow R_1 - 4R_2| \to \left(\begin{array}{ccc|c} 1 & 0 & 1 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right),$$

hence $x_1 + x_3 = 2$, $x_2 = 1$. Then the solution is $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_1 \\ 1 \\ 2-x_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} + x_1 \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$, $x_1 \in \mathbb{R}$.

It means the system (1) has infinitely many solutions in this case (iii).

<u>**Case 2**</u>. Let $\alpha = -1$. Then

$$\left(\begin{array}{ccc|c} 1 & 4 & \alpha & 6 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & (1-\alpha)(1+\alpha) & 2(1-\alpha) \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 4 & -1 & 6 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 4 \end{array}\right)$$

The third row yields $0 = 4$, which is false. It means the system (1) has no solution in this case (ii).

<u>Case 3</u>. Let $\alpha \neq 1$ and $\alpha \neq -1$. Then

$$\left(\begin{array}{ccc|c} 1 & 4 & \alpha & 6 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & (1-\alpha)(1+\alpha) & 2(1-\alpha) \end{array}\right) \to \left|R_3 \leftarrow \frac{R_3}{(1-\alpha)(1+\alpha)}\right| \to \left(\begin{array}{ccc|c} 1 & 4 & \alpha & 6 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & \frac{2(1-\alpha)}{((1-\alpha)(1+\alpha))} \end{array}\right) \to$$

$$\to \left(\begin{array}{ccc|c} 1 & 4 & \alpha & 6 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2/(1+\alpha) \end{array}\right) \to |R_1 \leftarrow R_1 - 4R_2| \to$$

$$\to \left(\begin{array}{ccc|c} 1 & 0 & \alpha & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2/(1+\alpha) \end{array}\right) \to |R_1 \leftarrow R_1 - \alpha R_3| \to \left(\begin{array}{ccc|c} 1 & 0 & 0 & 2 - 2\alpha/(1+\alpha) \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2/(1+\alpha) \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 0 & 0 & 2/(1+\alpha) \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2/(1+\alpha) \end{array}\right),$$

hence $x_1 = \frac{2}{1+\alpha}$, $x_2 = 1$, $x_3 = \frac{2}{1+\alpha}$. It means the system (1) has the unique solution in this case (i).

Answer:

(i) $\alpha \neq 1$ and $\alpha \neq -1$;
(ii) $\alpha = -1$;
(iii) $\alpha = 1$.