Answer on Question #60659 – Math – Differential Equations

Question

Solve

dy/dx =- y^3+3x^2y ÷ x^3+3xy^2

Solution

Method 1

Given

$$\frac{dy}{dx} = -\frac{y^3+3x^2y}{x^3+3xy^2}, \qquad (1)$$

divide the numerator and the denominator in the right-hand side of equation (1) by x^3

Then

$$\frac{dy}{dx} = -\frac{\left(\frac{y}{x}\right)^3+3\frac{y}{x}}{1+3\left(\frac{y}{x}\right)^2}. \qquad (2)$$

Substitute

$$u = u(x) = \frac{y}{x}, \qquad (3)$$

hence $y = ux, \frac{dy}{dx} = x\frac{du}{dx} + u$.

In terms of x and u equation (2) takes the following form:

$$x\frac{du}{dx} + u = -\frac{u^3+3u}{1+3u},$$

$$x\frac{du}{dx} = -\frac{u^3+3u}{1+3u} - u,$$

$$x\frac{du}{dx} = \frac{-u^3-3u-u-3u^2}{3u+1},$$

$$x\frac{du}{dx} = -u\frac{u^2+3u+4}{3u+1}$$

$$\frac{3u+1}{u(u^2+3u+4)} = -\frac{dx}{x} \qquad (4)$$

Equation $u^2 + 3u + 4 = 0$ does not have real solutions.

$$\frac{3u+1}{u(u^2+3u+4)} = \frac{A}{u} + \frac{Bu+C}{u^2+3u+4}$$

Reduce the right-hand side to the common denominator

$$\frac{3u+1}{u(u^2+3u+4)} = \frac{A(u^2+3u+4)+u(Bu+C)}{u(u^2+3u+4)} = \frac{(A+B)u^2+(3A+C)u+4A}{u(u^2+3u+4)}$$

Numerators in the left-hand and in the right-hand sides are equal:

$$3u+1=(A+B)u^2+(3A+C)u+4A\,.$$

Equate the like terms and get the following system of equations:

$$\begin{cases} A+B=0, \\ 3A+C=3, \\ 4A=1, \end{cases}$$

$$\begin{cases} B=-A, \\ C=3-3A, \\ A=\dfrac{1}{4}. \end{cases}$$

$$\begin{cases} B=-\dfrac{1}{4}, \\ C=3-\dfrac{3}{4}=\dfrac{9}{4}, \\ A=\dfrac{1}{4}. \end{cases}$$

Thus,

$$(u^2+3u+4)'=2u+3$$

$$\frac{3u+1}{u(u^2+3u+4)}=\frac{1}{4u}+\frac{-u+9}{4(u^2+3u+4)}=\frac{1}{4u}+\frac{-u-\frac{3}{2}+9+\frac{3}{2}}{4\left(u^2+2\cdot\frac{3}{2}u+\frac{9}{4}+4-\frac{9}{4}\right)}$$

$$=\frac{1}{4u}-\frac{u+\frac{3}{2}}{4\left(\left(u+\frac{3}{2}\right)^2+\frac{7}{4}\right)}+\frac{21}{2\cdot 4\left(\left(u+\frac{3}{2}\right)^2+\frac{7}{4}\right)}$$

$$=\frac{1}{4u}-\frac{1}{2\cdot 4}\cdot\frac{(u^2+3u+4)'}{(u^2+3u+4)}+\frac{21}{8}\cdot\frac{1}{\left(u+\frac{3}{2}\right)^2+\frac{7}{4}}$$

Integrating both sides of (4)

$$\int\left(\frac{1}{4u}-\frac{1}{8}\cdot\frac{(u^2+3u+4)'}{u^2+3u+4}+\frac{21}{8}\cdot\frac{1}{\left(u+\frac{3}{2}\right)^2+\frac{7}{4}}\right)du=-\int\frac{dx}{x}$$

$$\tfrac{1}{4}\,ln|u|-\tfrac{1}{8}\,ln(u^2+3u+4)+\tfrac{21}{8}\cdot\tfrac{1}{\frac{\sqrt{7}}{2}}\,tan^{-1}\tfrac{u+\frac{3}{2}}{\frac{\sqrt{7}}{2}}+C_1=-ln|x|,$$

where tan^{-1} is the inverse of the tangent, C_1 is an integration constant,

$$\tfrac{1}{4}\,ln|u|-\tfrac{1}{8}\,ln(u^2+3u+4)+\tfrac{3\sqrt{7}}{4}\,tan^{-1}\tfrac{2u+3}{\sqrt{7}}+ln|x|+C_1=0,$$

$$\tfrac{1}{4}\,ln\tfrac{|u|\cdot|x|^4}{\sqrt{u^2+3u+4}}+\tfrac{3\sqrt{7}}{4}\,tan^{-1}\tfrac{2u+3}{\sqrt{7}}+C_1=0, \qquad (5)$$

multiplying (5) by 4 and substituting (3) into (5) obtain

$$ln\frac{\left|\frac{y}{x}\right|\cdot|x|^4}{\sqrt{\left(\frac{y}{x}\right)^2+3\frac{y}{x}+4}}+3\sqrt{7}\cdot tan^{-1}\frac{2\frac{y}{x}+3}{\sqrt{7}}+C=0,$$

$$ln\frac{|y||x|^4}{\sqrt{y^2+3yx+4x^2}}+3\sqrt{7}\cdot tan^{-1}\frac{2y+3x}{x\sqrt{7}}+C=0$$

Method 2

$$\frac{dy}{dx}=-\frac{y^3+3x^2y}{x^3+3xy^2}, \qquad (6)$$

Rewrite the differential equation (6) in the following form:

$$\frac{dy}{dx}=-\frac{F_x}{F_y}, \qquad (7)$$

where $F_x=y^3+3x^2y, F_y=x^3+3xy^2$.

Equation (7) can be rewritten as

$$F_xdx+F_ydy=0. \qquad (8)$$

If the function $F=F(x,y)$ is such a function that

$\frac{\partial F}{\partial x}=F_x, \frac{\partial F}{\partial y}=F_y,$

then equation (8) becomes

$\frac{\partial F}{\partial x}dx+\frac{\partial F}{\partial y}dy=0,$

that is,

$dF(x,y)=0,$

hence $F(x,y)=C$, where C is an arbitrary real constant.

The rule for implicit differentiation in two variables is

$$\frac{dy}{dx}=-\frac{F_x}{F_y}$$

$F_x=(y^3+3x^2y)\rightarrow F=\int(y^3+3x^2y)dx=(xy^3+x^3y)+g(y),$

$F_y=x^3+3xy^2=x^3+3xy^2+g'(y),$

$g'(y)=0\rightarrow g(y)=C,$

where C is an integration constant,

$F(x,y)=xy^3+x^3y+C=0.$