Answer on Question #60655 - Math - Statistics and Probability

a) Let the random variable X have the following distribution:
P(X = 0) = P(X = 2) = p and P(X = 1) =1− 2 p where
0 ≤ p ≤1/2
for what value of p is the var(X) a maximum? Justify. (3)

Solution

i	1	2	3
X	0	1	2
P	p	1-2p	p

$$Var(X) = D(X) = M(X^2) - \left(M(X)\right)^2 = \sum_{i=1}^{3} x_i{}^2 p_i - \left(\sum_{i=1}^{3} x_i p_i\right)^2$$
$$= (0^2 \cdot p + 1^2 \cdot (1-2p) + 2^2 \cdot p) - (0 \cdot p + 1 \cdot (1-2p) + 2 \cdot p)^2$$
$$= 1 - 2p + 4p - 1 = 2p$$
$$Max\{Var(X)\} = Max\left\{2p, 0 \le p \le \frac{1}{2}\right\} = 1$$

Answer: $Max\{Var(X)\} = 1$

b) In a binomial distribution consisting of 5 independent trials, probabilities of 1 and 2 successes are 0.4096 and 0.2048 respectively. Find the parameter ‘ p ’ of the distribution. (3)

Solution

Probability mass function of a binomial distribution:
$$P_n(k) = \binom{n}{k} p^k (1-p)^{n-k}, where\ \binom{n}{k} = \frac{n!}{k!\,(n-k)!}$$
by task:
$$n = 5$$
$$P_5(1) = 0.4096 = \frac{5!}{1!\,(5-1)!} p^1 (1-p)^{5-1} = 5p(1-p)^4$$
$$P_5(2) = 0.2048 = \frac{5!}{2!\,(5-2)!} p^2 (1-p)^{5-2} = 10p^2(1-p)^3$$
To find the the parameter p we need to solve the appropriate system of equations:
$$\begin{cases} 5p(1-p)^4 = 0.4096 \\ 10p^2(1-p)^3 = 0.2048 \end{cases}$$

To solve this system divide the first equation by the second:

$$\frac{5p(1-p)^4}{10p^2(1-p)^3} = \frac{0.4096}{0.2048}$$

$$\frac{1-p}{2p} = 2$$

$$1-p = 4p$$

$$5p = 1$$

$$p = \frac{1}{5} = 0.2$$

Answer: $p = 0.2$

c) Fit a Poisson distribution to the following data: (4)

No. of mistakes per page:	0	1	2	3	4	Total
Number of pages:	109	65	22	3	1	200

Solution

Calculating the mean and variance:

$$E(N) = M(x) = \sum_{i=1}^{5} x_i p_i = \frac{1}{200}(0 \cdot 109 + 1 \cdot 65 + 2 \cdot 22 + 3 \cdot 3 + 4 \cdot 1) = 0.61$$

$$Var(X) = D(X) = M(X^2) - \left(M(X)\right)^2 = \sum_{i=1}^{3} {x_i}^2 p_i - \left(\sum_{i=1}^{3} x_i p_i\right)^2$$

$$= \frac{1}{200}(0^2 \cdot 109 + 1^2 \cdot 65 + 2^2 \cdot 22 + 3^2 \cdot 3 + 4^2 \cdot 1) - 0.61^2$$

$$= 0.98 - 0.37 = 0.61$$

Since $E(N) = \ Var(X)$, this can be modeled as a Poisson distribution because in this distribution $E(N) = \ Var(X) = \lambda$, and here we can assign $\lambda = 0.61$:

$$\Pr(N = k) = \frac{\lambda^k}{k!} e^{-\lambda} = \frac{0.61^k}{k!} e^{-0.61}, for\ k \geq 0$$

Answer: $\Pr(N = k) = \frac{0.61^k}{k!} e^{-0.61},\ for\ k \geq 0$