Answer on Question #60416 – Math – Abstract Algebra

Question

Prove that two conjugate sub-groups have the same order.

Solution

Let $H, P \subseteq X$ be conjugate sub-groups of group X. Then there exists $g \in X$ such that $P = g^{-1}Hg$. For any $h \in H$ there exists $p \in P$ for which $p = g^{-1}hg$.

Since isomorphic sub-groups have the same order, it remains for us to prove that $H \cong P$ (that is, subgroups H and P are isomorphic).

Consider the mapping $f : H \to P$ such that $f(h) = g^{-1}hg \in P$. Verify whether it is a homomorphism.

For any $h_1, h_2 \in H$ we have

$$f(h_1h_2) = g^{-1}h_1h_2g = g^{-1}h_1eh_2g = g^{-1}h_1(gg^{-1})h_2g = (g^{-1}h_1g)(g^{-1}h_2g) = f(h_1)f(h_2).$$

Hence f is a homomorphism.

Prove that f is a bijection.

Let $f(h_1) = f(h_2)$. Then

$$g^{-1}h_1g = g^{-1}h_2g \Rightarrow [multiply.by.g] \Rightarrow g(g^{-1}h_1g) = g(g^{-1}h_2g) \Rightarrow (gg^{-1})h_1g = (gg^{-1})h_2g \Rightarrow$$
$$\Rightarrow eh_1g = eh_2g \Rightarrow h_1g = h_2g$$

and

$$h_1g = h_2g \Rightarrow [multiply.by.g^{-1}] \Rightarrow (h_1g)g^{-1} = (h_2g)g^{-1} \Rightarrow h_1(gg^{-1}) = h_2(gg^{-1}) \Rightarrow h_1e = h_2e \Rightarrow h_1 = h_2.$$

We proved that $f(h_1) = f(h_2) \Rightarrow h_1 = h_2$, which means f is an injection.

An equality $P = g^{-1}Hg$ is equivalent to $gPg^{-1} = H$. For any $p \in P$ we have $gpg^{-1} \in H$ and

$$f(gpg^{-1}) = g^{-1}(gpg^{-1})g = (g^{-1}g)p(g^{-1}g) = epe = p.$$

Hence for every $p \in P$ there exists $h \in H (h = gpg^{-1})$ for which $f(h) = p$.

We proved that f is a surjection.

Hence f is a bijective homomorphism.

Thus, sub-groups H and P are isomorphic and have the same order.