Answer on Question #60412 – Math – Abstract Algebra

Question

Prove that two conjugates have the same order.

Proof

We need to prove that g and xgx^{-1} have the same order. It follows from the formula

$$(xgx^{-1})^n = xg^nx^{-1},$$

which shows $(xgx^{-1})^n = 1$ if and only if $g^n = 1$.

To see this, $(xgx^{-1})^n = 1$ implies

$\underbrace{xgx^{-1} \cdot xgx^{-1} \cdot xgx^{-1} \cdot \ldots \cdot xgx^{-1}xgx^{-1}}_{n} = xg(x^{-1}x)g(x^{-1}x)g \cdot \ldots \cdot g(x^{-1}x)gx^{-1} = xg1g1g \cdot \ldots \cdot$ $g1gx^{-1} = xg \cdot g \cdot g \cdot \ldots \cdot g \cdot g \cdot x^{-1} = xg^nx^{-1} = 1,$

hence $xg^n = x$,

that is, $g^n = 1$ by left-multiplying by x^{-1}.

The other direction of proof is clear.

It follows from this that the order of g divides the order of xgx^{-1} and vice versa, so the orders of g and xgx^{-1} must be equal.