Answer on Question #60155 – Math – Abstract Algebra

Question

Prove that center of a group is itself a subgroup.

Proof

Let us start from the definition of a center of a group.

<u>Definition</u> Let G be a group with respect to the operation $*$. Then the center of G is the subset
$$Z(G) \coloneqq \{z \in G \mid \forall g \in G,\ z * g = g * z\}.$$

Now we shall check the group axioms.

Closure

Let $z_1, z_2 \in Z(G)$. Then $z_1 * z_2 \in Z(G)$ because

$(z_1 * z_2) * g = z_1 * (z_2 * g) = z_1 * (g * z_2) = (z_1 * g) * z_2 = (g * z_1) * z_2 = g * (z_1 * z_2)$, because $z_1, z_2, g \in G$, besides, $z_1, z_2 \in Z(G)$.

Associativity

Obviously $(z_1 * z_2) * z_3 = z_1 * (z_2 * z_3)$ for all $z_1, z_2, z_3 \in Z(G)$ because $z_1, z_2, z_3 \in G$.

Identity element

The identity element $e \in G$ is also identity element of $Z(G)$ because $e * g = g * e = g$ for every $g \in G$.

Inverse element

The inverse element z^{-1} of the element $z \in Z(G)$ also belongs to $Z(G)$. Let us prove it. We start from the equality

$$z * g = g * z,\ z \in Z(G), g \in G.$$

Then we have

$z^{-1} * z * g = z^{-1} * g * z \Rightarrow g = z^{-1} * g * z \Rightarrow g * z^{-1} = z^{-1} * g$, that is, $z^{-1} * g = g * z^{-1}$, hence $z^{-1} \in Z(G)$ by definition of $Z(G)$.

For every $z \in Z(G)$ its inverse element z^{-1} also belongs to $Z(G)$.

All axioms are satisfied. This proves the statement.

Thus, the center of a group is itself a subgroup.