Answer on Question #60154 – Math – Differential Equations

Question

Find a homogeneous linear differential equation with constant coefficients whose general solution is given by

$$y(x) = c_1 + c_2 e^{2x}\cos 5x + c_3 e^{2x}\sin 5x.$$

Solution

General solution of a homogeneous linear differential equation with constant coefficients can be written as

$y(x) = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x} + c_3 e^{\lambda_3 x}$,

where λ_1, λ_2, λ_3 are the complex numbers.

The solution $y(x) = c_1 + c_2 e^{2x}\cos 5x + c_3 e^{2x}\sin 5x$ gives us the hint, that

$\lambda_1 = 0, \lambda_2 = 2 + 5i, \lambda_3 = 2 - 5i.$

Let's write the characteristic equation:

$(\lambda - \lambda_1)(\lambda - \lambda_2)(\lambda - \lambda_3) = 0.$

$$\begin{aligned}
&(\lambda - 0)\cdot(\lambda - (2+5i))\cdot(\lambda - (2-5i)) = \\
&= \lambda\cdot((\lambda-2) - 5i)\cdot((\lambda-2)+5i) = \\
&= \lambda\cdot((\lambda-2)^2 + 25) = \lambda\cdot(\lambda^2 - 4\lambda + 4 + 25) = \\
&= \lambda\cdot(\lambda^2 - 4\lambda + 4 + 25) = \lambda\cdot(\lambda^2 - 4\lambda + 29) = \\
&= \lambda^3 - 4\lambda^2 + 29\lambda = 0.
\end{aligned}$$

As a solution of any linear differential equation with constant coefficients can be found in the form of $y = e^{\lambda x}$, $y'(x) = \lambda e^{\lambda x}$, $y''(x) = \lambda^2 e^{\lambda x}$, $y'''(x) = \lambda^3 e^{\lambda x}$, it's obvious, that

$e^{\lambda x}\lambda^3 - 4e^{\lambda x}\lambda^2 + 29 e^{\lambda x}\lambda = 0$

and the differential equation is $y'''(x) - 4y''(x) + 29y'(x) = 0.$

Answer: $y'''(x) - 4y''(x) + 29y'(x) = 0.$