Answer on Question #60153 – Math – Differential Equations

Question

Find the general solution of $x^4y'' + x^3y' - 4x^2y = 1$, $x \in (0, \infty)$, given that $y_1 = x^2$ is a solution of the associated homogeneous equation.

Solution

The associated homogeneous equation is

$$x^4y'' + x^3y' - 4x^2y = 0 \qquad (1)$$

Its solutions can be found in the following form:

$$y = x^n$$

So

$$x^4y'' + x^3y' - 4x^2y = x^4n(n-1)x^{n-2} + x^3nx^{n-1} - 4x^2x^n = (n(n-1) + n - 4)x^{n+1} = 0$$

$$n(n-1) + n - 4 = 0 \to n = 2 \; and \; n = -2$$

Thus,
$y_1 = x^2$ and $y_2 = x^{-2}$ are solutions of (1).

Let $f_2(x) = x^4$, $f_1(x) = x^3$, $f_0(x) = -4x^2$, $g(x) = 1$, $f_2(x)y'' + f_1(x)y' + f_0(x)y = g(x)$,

$W(x) = y_1(x)\frac{d}{dx}y_2(x) - y_2(x)\frac{d}{dx}y_1(x) = x^2\frac{d}{dx}(x^{-2}) - x^{-2}\frac{d}{dx}x^2 = -\frac{2x^2}{x^3} - \frac{2x}{x^2} = -\frac{4}{x}$

The general solution of $x^4y'' + x^3y' - 4x^2y = 1$ is

$$y(x) = C_1y_1(x) + C_2y_2(x) + y_2(x)\int y_1(x)\frac{g(x)}{f_2(x)}\frac{dx}{W(x)} - y_1(x)\int y_2(x)\frac{g(x)}{f_2(x)}\frac{dx}{W(x)} =$$

$$= C_1x^2 + C_2x^{-2} + x^{-2}\int x^2\frac{1}{x^4}\frac{dx}{\left(-\frac{4}{x}\right)} - x^2\int x^{-2}\frac{1}{x^4}\frac{dx}{\left(-\frac{4}{x}\right)} =$$

$$= C_1x^2 + \frac{C_2}{x^2} - \frac{1}{4x^2}\int\frac{dx}{x} + \frac{x^2}{4}\int\frac{dx}{x^5} =$$

$= C_1x^2 + \frac{C_2}{x^2} - \frac{1}{4x^2}ln|x| + \frac{x^2}{4}\cdot\frac{x^{-5+1}}{-5+1} == C_1x^2 + \frac{C_2}{x^2} - \frac{1}{4x^2}ln|x| - \frac{1}{16x^2} = Cx^2 + \frac{1}{x^2}\left(D - \frac{1}{4}ln|x|\right)$, where C, D are arbitrary real constants.

Answer: $Cx^2 + \frac{1}{x^2}\left(D - \frac{1}{4}ln|x|\right)$.