Answer on Question #60152 – Math – Differential Equations

Question

Use the method of variation of parameters to find a particular solution of the following differential equation

$$y'' - y = \frac{e^x}{e^x + 1}$$

Solve the given differential equation

Solution

First we need to find the general solution to the homogeneous equation

$$y'' - y = 0.$$

The characteristic equation for this ODE is

$$\lambda^2 - 1 = 0 \Rightarrow \lambda = \pm 1.$$

Hence the general solution is

$$y_g(x) = c_1 e^x + c_2 e^{-x}.$$

We shall use the method of variation of parameters to find a solution of the form

$$y_g(x) = y_1(x)e^x + y_2(x)e^{-x}.$$

The variation of parameters formula gives the system

$$\begin{cases} y_1'(x)e^x + y_2'(x)e^{-x} = 0 \\ y_1'(x)e^x - y_2'(x)e^{-x} = \dfrac{e^x}{e^x + 1} \end{cases}.$$

Adding the equations obtain

$$2y_1'(x)e^x = \frac{e^x}{e^x + 1}$$

$$y_1'(x) = \frac{1}{2(e^x + 1)}$$

Integrating both sides with respect to x

$$y_1(x) = \int \frac{dx}{2(e^x + 1)} = \frac{1}{2}(x - \ln(e^x + 1)) + C_3.$$

Using the first equation $y_1'(x)e^x + y_2'(x)e^{-x} = 0$ of the system and plugging $y_1'(x) = \frac{1}{2(e^x+1)}$

gives

$$y_2'(x) = -y_1'(x)e^{2x} = -\frac{e^{2x}}{2(e^x + 1)}$$

Integrating both sides with respect to x

$$y_2(x) = \int\left(-\frac{e^{2x}}{2(e^x+1)}\right)dx = -\frac{1}{2}(e^x - \ln(e^x+1)) + C_4.$$

The general solution is

$$y_g(x) = \frac{e^x}{2}(x - \ln(e^x+1) + C_3) - \frac{e^{-x}}{2}(e^x - \ln(e^x+1) + C_4) =$$
$$= C\,e^x + De^{-x} + \frac{e^x}{2}(x - \ln(e^x+1)) - \frac{e^{-x}}{2}(e^x - \ln(e^x+1)).$$

Set $C = 0, D = 0.$

Hence a particular solution is given by

$$y_p(x) = \frac{e^x}{2}(x - \ln(e^x+1)) - \frac{e^{-x}}{2}(e^x - \ln(e^x+1)).$$

Answer:

$y_p(x) = \frac{e^x}{2}(x - \ln(e^x+1)) - \frac{e^{-x}}{2}(e^x - \ln(e^x+1));$

$y_g(x) = Ce^x + De^{-x} + \frac{e^x}{2}(x - \ln(e^x+1)) - \frac{e^{-x}}{2}(e^x - \ln(e^x+1)).$