Answer on Question #60151 – Math – Differential Equations

Question

Solve the following differential equation $y'' + y = cos^2x$.

Solution

The homogeneous second order linear differential equation is

$$y'' + y = 0. \quad (1)$$

It is a differential equation with constant coefficients.
The characteristic polynomial of the differential equation (1) is

$$r^2 + 1 = 0.$$

There are two complex conjugate roots:

$$r_1 = i \text{ and } r_2 = -i.$$

Hence the general solution of the homogeneous equation (1) is

$$y_h = C_1cosx + C_2sinx,$$

where C_1, C_2 are arbitrary real constants.
Using power-reducing/half angle formulas $cos^2x = \frac{1+\cos(2x)}{2}$ equation $y'' + y = cos^2x$ can be rewritten in the following form:

$$y'' + y = \frac{1}{2} + \frac{1}{2}cos(2x) \quad (2)$$

A particular solution of (2) will be sought in the form

$$y_p = A + Bcos(2x) + Csin(2x), \quad (3)$$

where A, B, C are arbitrary real constants.
Differentiating (3) with respect to *x* get

$$y_p' = -2Bsin(2x) + 2Ccos(2x)$$
$$y_p'' = -4Bcos(2x) - 4Csin(2x)$$

Substituting these derivatives in equation (2) get

$$-4Bcos(2x) - 4Csin(2x) + A + Bcos(2x) + Csin(2x) = \frac{1}{2} + \frac{1}{2}cos(2x)$$

Equating the corresponding coefficients obtain

$$\begin{cases} A = \frac{1}{2} \\ -3B = \frac{1}{2} \\ -3C = 0 \end{cases} \rightarrow \begin{cases} A = \frac{1}{2} \\ B = -\frac{1}{6} \\ C = 0 \end{cases}$$

Thus,

$$y_p = \frac{1}{2} - \frac{1}{6}cos(2x).$$

The general solution of the equation (2) is the sum of a particular solution y_p of the equation (2) and the general solution y_h of the homogeneous equation (1).
Therefore,

$$y = C_1cosx + C_2sinx + \frac{1}{2} - \frac{1}{6}cos(2x),$$

where C_1, C_2 are arbitrary real constants.

Answer: $y = C_1cosx + C_2sinx + \frac{1}{2} - \frac{1}{6}cos(2x)$.