Answer on Question #60150 – Math – Differential Equations

Question

Solve the linear system of differential equations

$$\begin{cases} \frac{dx}{dt} = x - 2y \\ \frac{dy}{dt} = x - y \end{cases}$$

Solution

Given

$$\begin{cases} \frac{dx}{dt} = x - 2y \\ \frac{dy}{dt} = x - y \end{cases} \qquad (1)$$

1) Equation $\frac{dy}{dt} = x - y$ from (1) gives

$$x = \frac{dy}{dt} + y \qquad (2)$$

2) Differentiate (2) with respect to t:

$$\frac{dx}{dt} = \frac{d^2y}{dt^2} + \frac{dy}{dt} \qquad (3)$$

3) Using (2) and (3) substitute for x and $\frac{dx}{dt}$ into the first equation $\frac{dx}{dt} = x - 2y$ of (1):

$$\frac{d^2y}{dt^2} + \frac{dy}{dt} = \frac{dy}{dt} + y - 2y,$$

$$\frac{d^2y}{dt^2} = -y.$$

The characteristic equation is

$$a^2 = -1,$$
$$a^2 + 1 = 0.$$

Its solutions are

$$a_1 = i\,, a_2 = -i,$$

hence

$$y(t) = C_1\cos(t) + C_2\sin(t) \qquad (4)$$

4) Differentiating (4) with respect to t::

$$\frac{dy}{dt} = -C_1 \sin(t) + C_2 \cos(t) \qquad (5)$$

Substituting (4), (5) into (2)

$$x(t)=\frac{dy}{dt}+y\ =-C_1\sin(t)+C_2\cos(t)+C_1\cos(t)+C_2\sin(t)=$$
$$=(C_2-C_1)\sin(t)+(C_1+C_2)\cos(t).$$

5) The general solution of system (1) is

$$\begin{cases}x(t)=(C_2-C_1)\sin(t)+(C_1+C_2)\cos(t)\,,\\ \qquad\; y(t)=C_1\cos(t)+C_2\sin(t)\,,\end{cases}\quad C_1,C_2 \text{ are arbitrary real constants.}$$

Answer:

$$\begin{cases}x(t)=(C_2-C_1)\sin(t)+(C_1+C_2)\cos(t)\,,\\ \qquad\; y(t)=C_1\cos(t)+C_2\sin(t)\,,\end{cases}\quad C_1,C_2 \text{ are arbitrary real constants.}$$