Answer on Question #59928 – Math – Abstract Algebra

Question

Prove that a non-commutative group has at least six elements.

Proof

Lagrange's theorem shows that any group of prime order is cyclic and therefore commutative. It is obvious that

1. If group has 1 element, it is commutative: $\mathbb{G} = \{e\}: ee = e$.
2. If group has 2 elements, it is cyclic of order 2 and commutative:
   $\mathbb{G} = \{e, a\}$: $a^2 = e,\ a = a^{-1},\ aa^{-1} = e = a^{-1}a.$
3. If group has 3 elements it is cyclic of order 3 and commutative:
   $\mathbb{G} = \{e, a, b\}$: $a^3 = e, b = a^{-1} = a^2, ab = ba = e.$
4. If group has 4 elements, it is isomorphic
   either to
   a) cyclic group of order 4: $\mathbb{G} = \{e, a, b, c\}$: $a^4 = e, b = a^2, c = a^3\ a^3 = a^{-1}$, $ab = ba = c, ac = ca = e, bc = cb = a$;

   or to

   b) Klein four-group: $\mathbb{G} = \{e, a, b, c\}$: $a^2 = b^2 = c^2 = e,\ ab = ba = c,\ ac = ca = b$, $bc = cb = a.$
5. If group has 5 elements, it is isomorphic to the cyclic group of order 5, which is commutative:
   $\mathbb{G} = \{e, a, a^2, a^3, a^4\}$.

Thus, if group has 5 or less elements, there is no possibility to define a non-commutative group operation.

If there are 6 or more elements, we have "enough space" to introduce a non-commutative group operation. The symmetric group S_3 has 6 elements and it is non-commutative, because commutativity does not hold for all elements of S_3.

For example,

$(1\ 2)(1\ 3) = (1\ 3\ 2), (1\ 3)(1\ 2) = (1\ 2\ 3)$, that is, $(1\ 2)(1\ 3) \neq (1\ 3)(1\ 2)$ (non-commutative elements);

though $(1\ 2\ 3)(1\ 3\ 2) = e$, $(1\ 3\ 2)(1\ 2\ 3) = e$, that is, $(1\ 2\ 3)(1\ 3\ 2) = (1\ 3\ 2)(1\ 2\ 3)$ (commutative elements).

Answer: we have proved that group has to have at least 6 elements to be a non-commutative one.