Answer on Question #59927 – Math – Abstract Algebra

Question

Prove that every group of prime order is cyclic and hence abelian

Proof

Let p be a prime and G be a group such that |G|=p. Then G contains more than one element such that $g \neq e_G$ and $\langle g \rangle$ contains more than one element.

By the Lagrange's theorem, if $\langle g \rangle \leq G$ then the order of any element in a group divides the p. If a group has a prime order, than effectively the order of any non-identity element must equal the order of the group (since it can't be 1). And the group therefore has a generator.

Since $|\langle g \rangle| > 1$ and $|\langle g \rangle|$ divides a prime $|\langle g \rangle| = p$. Hence $\langle g \rangle$=G. So, it is cyclic.

Thus, every group of prime order is cyclic.

G is called a commutative(or abelian) group if $\forall a, b \in G \ \ a * b = b * a$.

"G is cyclic" means there is $g \in G$ such that for every $x \in G$ there exists $n \in Z$ such that $x = g^n$. If $a, b \in G$ then there exists $k, m \in Z$ such that $a = g^k$, $b = g^m$.

Next,

$$ab = g^k g^m = g^{k+m} = g^{m+k} = g^m g^k = ba$$

So, G is abelian.

Thus, every cyclic group is abelian.