Answer on Question #59824 – Math – Linear Algebra

Question

Find the solution to the following linear system by using

(I) Cramer rules

(II) Inverse of matrix

(III) Math Lab with two different methods

$$\begin{cases} X + 2Y - Z = 2, \\ 3X + Y + Z = 4, \\ X - Y + Z = 6. \end{cases}$$

Solution

(I) Using the Cramer rule

$$\Delta = \begin{vmatrix} 1 & 2 & -1 \\ 3 & 1 & 1 \\ 1 & -1 & 1 \end{vmatrix} = 1 \cdot \begin{vmatrix} 1 & 1 \\ -1 & 1 \end{vmatrix} - 2 \begin{vmatrix} 3 & 1 \\ 1 & 1 \end{vmatrix} + (-1) \cdot \begin{vmatrix} 3 & 1 \\ 1 & -1 \end{vmatrix} = 1 + 1 - 2(3 - 1) -$$

$$-(-3 - 1) = 2 - 4 + 4 = 2,$$

$$\Delta_X = \begin{vmatrix} 2 & 2 & -1 \\ 4 & 1 & 1 \\ 6 & -1 & 1 \end{vmatrix} = 2 \cdot 1 \cdot 1 + 2 \cdot 1 \cdot 6 + (-1) \cdot 4 \cdot (-1) - 6 \cdot 1 \cdot (-1) - 1 \cdot 4 \cdot 2 - (-1) \cdot$$

$$1 \cdot 2 = 2 + 12 + 4 + 6 - 8 + 2 = 18,$$

$$\Delta_Y = \begin{vmatrix} 1 & 2 & -1 \\ 3 & 4 & 1 \\ 1 & 6 & 1 \end{vmatrix} = 1 \cdot 4 \cdot 1 + 2 \cdot 1 \cdot 1 + (-1) \cdot 3 \cdot 6 - 1 \cdot 4 \cdot (-1) - 6 \cdot 1 \cdot 1 - 1 \cdot 3 \cdot 2 = 4 +$$

$$2 - 18 + 4 - 6 - 6 = -20,$$

$$\Delta_Z = \begin{vmatrix} 1 & 2 & 2 \\ 3 & 1 & 4 \\ 1 & -1 & 6 \end{vmatrix} = 1 \cdot 1 \cdot 6 + 2 \cdot 4 \cdot 1 + 2 \cdot 3 \cdot (-1) - 1 \cdot 1 \cdot 2 - (-1) \cdot 4 \cdot 1 - 6 \cdot 2 \cdot 3 = 6 +$$

$$+8 - 6 - 2 + 4 - 36 = -26.$$

The solution is

$$X = \frac{\Delta_X}{\Delta} = \frac{18}{2} = 9,\ Y = \frac{\Delta_Y}{\Delta} = \frac{-20}{2} = -10,\ Z = \frac{\Delta_Z}{\Delta} = \frac{-26}{2} = -13.$$

(II) We write the system of equations in the form

$$A\tilde{X} = B,$$

where $A = \begin{pmatrix} 1 & 2 & -1 \\ 3 & 1 & 1 \\ 1 & -1 & 1 \end{pmatrix}$, $\tilde{X} = \begin{pmatrix} X \\ Y \\ Z \end{pmatrix}$ and $B = \begin{pmatrix} 2 \\ 4 \\ 6 \end{pmatrix}$.

Multiplying the equation $A\tilde{X} = B$ by the matrix A^{-1} on the left get

$$A^{-1}A\tilde{X} = A^{-1}B \rightarrow \tilde{X} = A^{-1}B$$

The inverse matrix A^{-1} of A is given by

$$A^{-1} = \frac{1}{\Delta} \begin{pmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{pmatrix},$$

where

Δ is the determinant of the matrix A;

A_{ij} is the cofactor of the matrix element a_{ij};

$$\Delta = \begin{vmatrix} 1 & 2 & -1 \\ 3 & 1 & 1 \\ 1 & -1 & 1 \end{vmatrix} = 1 \cdot 1 \cdot 1 + 2 \cdot 1 \cdot 1 + (-1) \cdot 3 \cdot (-1) - 1 \cdot 1 \cdot (-1) - 1 \cdot 1 \cdot (-1) -$$

$-1 \cdot 2 \cdot 3 = 1 + 2 + 3 + 1 + 1 - 6 = 2;$

$A_{11} = (-1)^{1+1} \begin{vmatrix} 1 & 1 \\ -1 & 1 \end{vmatrix} = 1 + 1 = 2;$ $A_{12} = (-1)^{1+2} \begin{vmatrix} 3 & 1 \\ 1 & 1 \end{vmatrix} = -(3 - 1) = -2;$

$A_{13} = (-1)^{1+3} \begin{vmatrix} 3 & 1 \\ 1 & -1 \end{vmatrix} = -3 - 1 = -4;$ $A_{21} = (-1)^{2+1} \begin{vmatrix} 2 & -1 \\ -1 & 1 \end{vmatrix} = -(2 - 1) = -1;$

$A_{22} = (-1)^{2+2} \begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix} = 1 + 1 = 2;$ $A_{23} = (-1)^{2+3} \begin{vmatrix} 1 & 2 \\ 1 & -1 \end{vmatrix} = -(-1 - 2) = 3;$

$A_{31} = (-1)^{3+1} \begin{vmatrix} 2 & -1 \\ 1 & 1 \end{vmatrix} = 2 + 1 = 3;$ $A_{32} = (-1)^{3+2} \begin{vmatrix} 1 & -1 \\ 3 & 1 \end{vmatrix} = -(1 + 3) = -4;$

$A_{33} = (-1)^{3+3} \begin{vmatrix} 1 & 2 \\ 3 & 1 \end{vmatrix} = 1 - 6 = -5.$

Hence

$$A^{-1} = \frac{1}{2} \begin{pmatrix} 2 & -1 & 3 \\ -2 & 2 & -4 \\ -4 & 3 & -5 \end{pmatrix} = \begin{pmatrix} 1 & -\frac{1}{2} & \frac{3}{2} \\ -1 & 1 & -2 \\ -2 & \frac{3}{2} & -\frac{5}{2} \end{pmatrix}.$$

The solution is given by

$$\begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = A^{-1}B = \begin{pmatrix} 1 & -\frac{1}{2} & \frac{3}{2} \\ -1 & 1 & -2 \\ -2 & \frac{3}{2} & -\frac{5}{2} \end{pmatrix} \begin{pmatrix} 2 \\ 4 \\ 6 \end{pmatrix} = \begin{pmatrix} 1 \cdot 2 - \frac{1}{2} \cdot 4 + \frac{3}{2} \cdot 6 \\ -1 \cdot 2 + 1 \cdot 4 + (-2) \cdot 6 \\ -2 \cdot 2 + \frac{3}{2} \cdot 4 - \frac{5}{2} \cdot 6 \end{pmatrix} = \begin{pmatrix} 2 - 2 + 9 \\ -2 + 4 - 12 \\ -4 + 6 - 15 \end{pmatrix} =$$

$$= \begin{pmatrix} 9 \\ -10 \\ -13 \end{pmatrix}.$$

$X = 9, Y = -10, Z = -13.$

(III) Using MatLab solution of the system can be obtained by means of the inverse matrix:

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  To get started, select "MATLAB Help" from the Help menu.

>> A=[1 2 -1;3 1 1;1 -1 1]

A =

     1     2    -1
     3     1     1
     1    -1     1

>> B=[2;4;6]

B =

     2
     4
     6

>> inv(A)*B

ans =

    9.0000
  -10.0000
  -13.0000

>> |
```

We solve the system of equations by Gauss

```
>> A=[1 2 -1;3 1 1;1 -1 1]; B=[2;4;6]; C=[A B];
>> D=rref(C)

D =

     1     0     0     9
     0     1     0   -10
     0     0     1   -13

>>
```

The solution of the system of the last column of the matrix D

Answer: $X = 9,\ Y = -10\ ,\ Z = -13.$