ANSWER on Question #59207 – Math – Calculus

QUESTION #1

If $\varphi(x,y,z) = 2xz^4 - x^2y$

Find $|\nabla\varphi(x,y,z)|$

SOLUTION

By the definition

$$\nabla = \vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z} \Leftrightarrow \nabla\varphi = \vec{i}\frac{\partial\varphi}{\partial x} + \vec{j}\frac{\partial\varphi}{\partial y} + \vec{k}\frac{\partial\varphi}{\partial z}$$

$$\varphi(x,y,z) = 2xz^4 - x^2y$$

$$\frac{\partial\varphi}{\partial x} = \frac{\partial}{\partial x}(2xz^4 - x^2y) = 2z^4 - 2xy$$

$$\frac{\partial\varphi}{\partial y} = \frac{\partial}{\partial y}(2xz^4 - x^2y) = -x^2$$

$$\frac{\partial\varphi}{\partial z} = \frac{\partial}{\partial z}(2xz^4 - x^2y) = 2x * 4z^3 = 8xz^3$$

$$\nabla\varphi = \vec{i}\frac{\partial\varphi}{\partial x} + \vec{j}\frac{\partial\varphi}{\partial y} + \vec{k}\frac{\partial\varphi}{\partial z} = \vec{i}(2z^4 - 2xy) + \vec{j}(-x^2) + \vec{k}(8xz^3) \Leftrightarrow$$

$$|\nabla\varphi(x,y,z)| = \sqrt{(2z^4 - 2xy)^2 + (-x^2)^2 + (8xz^3)^2}.$$

ANSWER: $|\nabla\varphi(x,y,z)| = \sqrt{(2z^4 - 2xy)^2 + (-x^2)^2 + (8xz^3)^2}.$

QUESTION #2

If $\varphi(x,y,z) = 3x^2y - y^3z^2$

find $\nabla\varphi$ at point (1,-2,-1)

SOLUTION

By the definition

$$\nabla = \vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z} \Leftrightarrow \nabla\varphi = \vec{i}\frac{\partial\varphi}{\partial x} + \vec{j}\frac{\partial\varphi}{\partial y} + \vec{k}\frac{\partial\varphi}{\partial z}$$

$$\varphi(x,y,z) = 3x^2y - y^3z^2$$

$$\frac{\partial \varphi}{\partial x} = \frac{\partial}{\partial x}(3x^2y - y^3z^2) = 3y * 2x = 6xy$$

$$\frac{\partial \varphi}{\partial y} = \frac{\partial}{\partial y}(3x^2y - y^3z^2) = 3x^2 - 3y^2z^2$$

$$\frac{\partial \varphi}{\partial z} = \frac{\partial}{\partial z}(3x^2y - y^3z^2) = -y^3 * 2z = -2y^3z$$

$$\nabla\varphi = \vec{\imath}\frac{\partial \varphi}{\partial x} + \vec{\jmath}\frac{\partial \varphi}{\partial y} + \vec{k}\frac{\partial \varphi}{\partial z} = \vec{\imath}(6xy) + \vec{\jmath}(3x^2 - 3y^2z^2) + \vec{k}(-2y^3z)$$

$$\nabla\varphi(1,-2,-1) = \left(\vec{\imath}(6xy) + \vec{\jmath}(3x^2 - 3y^2z^2) + \vec{k}(-2y^3z)\right)(1,-2,-1) =$$

$$= \vec{\imath}(6 * 1 * (-2)) + \vec{\jmath}(3 * (1)^2 - 3 * (-2)^2 * (-1)^2) + \vec{k}(-2 * (-2)^3 * (-1)) =$$

$$= \vec{\imath}(-12) + \vec{\jmath}(3 - 3 * 4 * 1) + \vec{k}(-2 * (-8) * (-1)) =$$

$$= \vec{\imath}(-12) + \vec{\jmath}(3 - 12) + \vec{k}(-16) = -12\vec{\imath} - 9\vec{\jmath} - 16\vec{k}.$$

ANSWER: $\nabla\varphi(1,-2,-1) = -12\vec{\imath} - 9\vec{\jmath} - 16\vec{k}.$

QUESTION #3

Find a unit normal to the surface $x^2y + 2xz = 4$ at point (2,-2,3).

SOLUTION

By definition, the unit normal to the surface $\varphi(x, y, z) = 0$ is

$$\vec{n} = \frac{\nabla\varphi}{|\nabla\varphi|}$$

The reference surface $\varphi(x, y, z)$ is defined by the equation

$$\varphi(x, y, z) = x^2y + 2xz - 4$$

By the definition

$$\nabla = \vec{\imath}\frac{\partial}{\partial x} + \vec{\jmath}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z} \Leftrightarrow \nabla\varphi = \vec{\imath}\frac{\partial \varphi}{\partial x} + \vec{\jmath}\frac{\partial \varphi}{\partial y} + \vec{k}\frac{\partial \varphi}{\partial z}$$

$$\frac{\partial \varphi}{\partial x} = \frac{\partial}{\partial x}(x^2y + 2xz - 4) = 2x * y + 2z = 2xy + 2z$$

$$\frac{\partial \varphi}{\partial y} = \frac{\partial}{\partial y}(x^2y + 2xz - 4) = x^2$$

$$\frac{\partial\varphi}{\partial z} = \frac{\partial}{\partial z}(x^2y + 2xz - 4) = 2x$$

$$\nabla\varphi = \vec{\imath}\frac{\partial\varphi}{\partial x} + \vec{\jmath}\frac{\partial\varphi}{\partial y} + \vec{k}\frac{\partial\varphi}{\partial z} = \vec{\imath}(2xy + 2z) + \vec{\jmath}(x^2) + \vec{k}(2x) \Leftrightarrow$$

$$\nabla\varphi(2,-2,3) = \left(\vec{\imath}(2xy + 2z) + \vec{\jmath}(x^2) + \vec{k}(2x)\right)(2,-2,3) =$$

$$= \vec{\imath}(2 * 2 * (-2) + 2 * 3) + \vec{\jmath}(2^2) + \vec{k}(2 * 2) = \vec{\imath}(-8 + 6) + 4\vec{\jmath} + 4\vec{k} =$$

$$= -2\vec{\imath} + 4\vec{\jmath} + 4\vec{k}$$

$$|\nabla\varphi| = \sqrt{(-2)^2 + 4^2 + 4^2} = \sqrt{4 + 16 + 16} = \sqrt{36} = 6$$

Hence, the desired unit normal looks

$$\vec{n} = \frac{\nabla\varphi}{|\nabla\varphi|} = \frac{1}{6}\left(-2\vec{\imath} + 4\vec{\jmath} + 4\vec{k}\right) = \frac{-1}{3}\vec{\imath} + \frac{2}{3}\vec{\jmath} + \frac{2}{3}\vec{k} =$$

$$= \frac{-1}{3}\vec{\imath} + \frac{2}{3}\vec{\jmath} + \frac{2}{3}\vec{k}.$$

ANSWER: $\vec{n} = \frac{-1}{3}\vec{\imath} + \frac{2}{3}\vec{\jmath} + \frac{2}{3}\vec{k}$.

QUESTION #4

Let $\varphi(x, y, z) = xy^2z$

and $\vec{A} = xz\vec{\imath} - xy^2\vec{\jmath} + yz^2\vec{k}$

Find

$$\frac{\partial^3}{\partial^2 x\partial z}(\varphi\vec{A})$$

SOLUTION

First you transform expression

$$\varphi\vec{A} = xy^2z * \left(xz\vec{\imath} - xy^2\vec{\jmath} + yz^2\vec{k}\right) =$$

$$= (xy^2z * xz)\vec{\imath} - (xy^2z * xy^2)\vec{\jmath} + (xy^2z * yz^2)\vec{k} = x^2y^2z^2\vec{\imath} - x^2y^4z\vec{\jmath} + xy^3z^3\vec{k}$$

We now turn to the calculation of the expression

$$\frac{\partial^3}{\partial^2 x \partial z}\left(\varphi\vec{A}\right) = \frac{\partial^3}{\partial^2 x \partial z}\left(x^2y^2z^2\vec{\imath} - x^2y^4z\vec{\jmath} + xy^3z^3\vec{k}\right) =$$

$$= \vec{\imath}\frac{\partial^3}{\partial^2 x \partial z}(x^2y^2z^2) + \vec{\jmath}\frac{\partial^3}{\partial^2 x \partial z}(-x^2y^4z) + \vec{k}\frac{\partial^3}{\partial^2 x \partial z}(xy^3z^3)$$

$$\frac{\partial^3}{\partial^2 x \partial z}(x^2y^2z^2) = \frac{\partial^2}{\partial^2 x}(2x^2y^2z) = \frac{\partial}{\partial x}(2 * 2xy^2z) = 4y^2z$$

$$\frac{\partial^3}{\partial^2 x \partial z}(-x^2y^4z) = \frac{\partial^2}{\partial^2 x}(-x^2y^4) = \frac{\partial}{\partial x}(-2xy^4) = -2y^4$$

$$\frac{\partial^3}{\partial^2 x \partial z}(xy^3z^3) = \frac{\partial^2}{\partial^2 x}(xy^3 * 3z^2) = \frac{\partial}{\partial x}(y^3 * 3z^2) = 0$$

Substitute all found the partial derivatives in the expression

$$\frac{\partial^3}{\partial^2 x \partial z}\left(\varphi\vec{A}\right) = \vec{\imath}\frac{\partial^3}{\partial^2 x \partial z}(x^2y^2z^2) + \vec{\jmath}\frac{\partial^3}{\partial^2 x \partial z}(-x^2y^4z) + \vec{k}\frac{\partial^3}{\partial^2 x \partial z}(xy^3z^3) =$$

$$= \vec{i} * 4y^2z + \vec{j} * (-2y^4z) + \vec{k} * 0 = 4y^2z * \vec{i} - 2y^4z * \vec{j}$$

ANSWER: $\frac{\partial^3}{\partial^2 x \partial z}\left(\varphi\vec{A}\right) = 4y^2z * \vec{\imath} - 2y^4z * \vec{\jmath}$.