ANSWER on Question №59206, Math / Differential Equations

QUESTION 6

If

$$\overrightarrow{A} = 5t^2\overrightarrow{i} + t\overrightarrow{j} - t^3\overrightarrow{k} \quad \text{and} \quad \overrightarrow{B} = \overrightarrow{i}\sin t - \overrightarrow{j}\cos t.$$

Evaluate

$$\frac{d}{dt}\left(\overrightarrow{A}\times\overrightarrow{B}\right)$$

SOLUTION

By the definition

$$\overrightarrow{A}\times\overrightarrow{B} = \left\| \begin{array}{ccc} \overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{array} \right\| = \overrightarrow{i}\left(A_yB_z - A_zB_y\right) - \overrightarrow{j}\left(A_xB_z - A_zB_x\right) + \overrightarrow{k}\left(A_xB_y - A_yB_x\right)$$

In our case

$$\overrightarrow{A} = \underbrace{5t^2}_{A_x}\overrightarrow{i} + \underbrace{t}_{A_y}\overrightarrow{j} + \underbrace{-t^3}_{A_z}\overrightarrow{k} \quad \text{and} \quad \overrightarrow{B} = \overrightarrow{i}\underbrace{\sin t}_{B_x} + \overrightarrow{j}\underbrace{-\cos t}_{B_y} + \underbrace{0}_{B_z}\overrightarrow{k}$$

$$\overrightarrow{A}\times\overrightarrow{B} = \overrightarrow{i}\left(t*0-(-t^3)*(-\cos t)\right) - \overrightarrow{j}\left((5t^2)*0-(-t^3)*\sin t\right) + \overrightarrow{k}\left((5t^2)*(-\cos t)-t*\sin t\right) =$$

$$= -t^3\cos t\,\overrightarrow{i} - t^3*\sin t\,\overrightarrow{j} - \left(5t^2\cos t + t*\sin t\right)\overrightarrow{k}$$

$$\frac{d}{dt}\left(\overrightarrow{A}\times\overrightarrow{B}\right) = \frac{d}{dt}\left(-t^3\cos t\,\overrightarrow{i} - t^3*\sin t\,\overrightarrow{j} - \left(5t^2\cos t + t*\sin t\right)\overrightarrow{k}\right) =$$

$$= \frac{d}{dt}\left(-t^3\cos t\right)\overrightarrow{i} - \frac{d}{dt}\left(t^3*\sin t\right)\overrightarrow{j} - \frac{d}{dt}\left(5t^2\cos t + t*\sin t\right)\overrightarrow{k} =$$

$$= \left(-3t^2\cos t + (-t^3)*(-\sin t)\right)\overrightarrow{i} - \left(3t^2*\sin t + t^3*\cos t\right)\overrightarrow{j} -$$

$$-\left(5*2t\cos t + 5t^2*(-\sin t) + \sin t + t*\cos t\right)\overrightarrow{k} =$$

$$= \left(t^3\sin t - 3t^2\cos t\right)\overrightarrow{i} - \left(3t^2\sin t + t^3\cos t\right)\overrightarrow{j} - \left(11t\cos t - 5t^2\sin t + \sin t\right)\overrightarrow{k}$$

ANSWER

$$\frac{d}{dt}\left(\overrightarrow{A}\times\overrightarrow{B}\right) = \left(t^3\sin t - 3t^2\cos t\right)\overrightarrow{i} - \left(3t^2\sin t + t^3\cos t\right)\overrightarrow{j} - \left(11t\cos t - 5t^2\sin t + \sin t\right)\overrightarrow{k}$$

QUESTION 7

If

$$\overrightarrow{A} = 5t^2\overrightarrow{i} + t\overrightarrow{j} - t^3\overrightarrow{k} \quad \text{and} \quad \overrightarrow{B} = \overrightarrow{i}\sin t - \overrightarrow{j}\cos t.$$

Evaluate

$$\frac{d}{dt}\left(\overrightarrow{A}\cdot\overrightarrow{A}\right)$$

SOLUTION

By the definition

$$\overrightarrow{A}\cdot\overrightarrow{A} = A_x * A_x + A_y * A_y + A_z * A_z$$

In our case

$$\overrightarrow{A} = \underbrace{5t^2}_{A_x}\overrightarrow{i} + \underbrace{t}_{A_y}\overrightarrow{j} + \underbrace{-t^3}_{A_z}\overrightarrow{k}$$

$$\overrightarrow{A}\cdot\overrightarrow{A} = 5t^2 * 5t^2 + t * t + (-t^3) * (-t^3) = 25t^4 + t^2 + t^6$$

$$\frac{d}{dt}\left(\overrightarrow{A}\cdot\overrightarrow{A}\right) = \frac{d}{dt}\left(25t^4 + t^2 + t^6\right) = 25 * 4t^3 + 2t + 6t^5 = 100t^3 + 2t + 6t^5$$

ANSWER

$$\frac{d}{dt}\left(\overrightarrow{A}\cdot\overrightarrow{A}\right) = 6t^5 + 100t^3 + 2t$$

QUESTION 8

If

$$\overrightarrow{A} = \sin u\,\overrightarrow{i} + \cos u\,\overrightarrow{j} + u\,\overrightarrow{k} \quad \overrightarrow{B} = \cos u\,\overrightarrow{i} - \sin u\,\overrightarrow{j} - 3\,\overrightarrow{k} \quad \text{and} \quad \overrightarrow{C} = 2\,\overrightarrow{i} + 3\,\overrightarrow{j} - \overrightarrow{k}$$

Evaluate

$$\frac{d}{du}\left(\overrightarrow{A}\times(\overrightarrow{B}\times\overrightarrow{C})\right) \quad \text{at} \quad u = 0$$

SOLUTION

By the definition

$$\overrightarrow{B}\times\overrightarrow{C} = \left\| \begin{matrix} \overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k} \\ \cos u & -\sin u & -3 \\ 2 & 3 & -1 \end{matrix} \right\| =$$

$$= \overrightarrow{i}((-\sin u)*(-1)-(3)*(-3)) - \overrightarrow{j}(\cos u*(-1)-2*(-3)) + \overrightarrow{k}(\cos u*(3)-2*(-\sin u)) =$$

$$= \overrightarrow{i}(\sin u + 9) - \overrightarrow{j}(6 - \cos u) + \overrightarrow{k}(3\cos u + 2\sin u) =$$

$$= \overrightarrow{i}(\sin u + 9) + \overrightarrow{j}(\cos u - 6) + \overrightarrow{k}(3\cos u + 2\sin u)$$

$$\overrightarrow{A}\times\left(\overrightarrow{B}\times\overrightarrow{C}\right) = \left\| \begin{matrix} \overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k} \\ \sin u & \cos u & u \\ \sin u + 9 & \cos u - 6 & 3\cos u + 2\sin u \end{matrix} \right\| =$$

$$= \overrightarrow{i}(\cos u*(3\cos u+2\sin u) - u*(\cos u - 6)) - \overrightarrow{j}(\sin u*(3\cos u + 2\sin u) - u*(\sin u + 9)) +$$

$$+\overrightarrow{k}(\sin u*(\cos u-6)-\cos u*(\sin u+9))=$$

$$=\overrightarrow{i}(3\cos^2 u+2\cos u\sin u-u\cos u+6u)-\overrightarrow{j}(3\sin u\cos u+2\sin^2 u-u\sin u-9u)+$$

$$+\overrightarrow{k}(\sin u\cos u-6\sin u-\cos u\sin u-9\cos u)=$$

$$=\overrightarrow{i}(3\cos^2 u+2\cos u\sin u-u\cos u+6u)-\overrightarrow{j}(3\sin u\cos u+2\sin^2 u-u\sin u-9u)-$$

$$-\overrightarrow{k}(6\sin u+9\cos u)$$

$$\frac{d}{du}\Big(\overrightarrow{A}\times(\overrightarrow{B}\times\overrightarrow{C})\Big)=$$

$$=\overrightarrow{i}\frac{d}{du}\big(3\cos^2 u+2\cos u\sin u-u\cos u+6u\big)-\overrightarrow{j}\frac{d}{du}\big(3\sin u\cos u+2\sin^2 u-u\sin u-9u\big)-$$

$$-\overrightarrow{k}\frac{d}{du}\big(6\sin u+9\cos u\big)=$$

$$=\overrightarrow{i}(3*2\cos u(-\sin u)+2(-\sin u)\sin u+2\cos u*\cos u-\cos u-u*(-\sin u)+6)-$$

$$-\overrightarrow{j}(3\cos u\cos u+3\sin u*(-\sin u)+2*2\sin u*\cos u-\sin u-u\cos u-9)-$$

$$-\overrightarrow{k}(6\cos u+9(-\sin u))=$$

$$=\overrightarrow{i}(-6\cos u\sin u-2\sin^2 u+2\cos^2 u-\cos u+u\sin u+6)-$$

$$-\overrightarrow{j}(3\cos^2 u-3\sin^2 u+4\sin u\cos u-\sin u-u\cos u-9)-\overrightarrow{k}(6\cos u-9\sin u)$$

$$\frac{d}{du}\Big(\overrightarrow{A}\times(\overrightarrow{B}\times\overrightarrow{C})\Big)\Big|_{u=0}=$$

$$=\Big(\overrightarrow{i}(-6\cos u\sin u-2\sin^2 u+2\cos^2 u-\cos u+u\sin u+6)-$$

$$-\overrightarrow{j}(3\cos^2 u-3\sin^2 u+4\sin u\cos u-\sin u-u\cos u-9)-\overrightarrow{k}(6\cos u-9\sin u)\Big)\Big|_{u=0}=$$

$$=\overrightarrow{i}(-6\cos 0\sin 0-2\sin^2 0+2\cos^2 0-\cos 0+0*\sin 0+6)-$$

$$-\overrightarrow{j}(3\cos^2 0-3\sin^2 0+4\sin 0\cos 0-\sin 0-0*\cos 0-9)-\overrightarrow{k}(6\cos 0-9\sin 0)=$$

$$=\overrightarrow{i}(2-1+6)-\overrightarrow{j}(3-9)-\overrightarrow{k}(6)=7\overrightarrow{i}+6\overrightarrow{j}-6\overrightarrow{k}$$

ANSWER

$$\frac{d}{du}\Big(\overrightarrow{A}\times(\overrightarrow{B}\times\overrightarrow{C})\Big)\Big|_{u=0}=7\overrightarrow{i}+6\overrightarrow{j}-6\overrightarrow{k}$$

QUESTION 9

Let

$$\overrightarrow{A}=x^2yz\overrightarrow{i}-2xz^3\overrightarrow{j}-xz^2\overrightarrow{k}\quad\text{and}\quad\overrightarrow{B}=4z\overrightarrow{i}+y\overrightarrow{j}+4x^2\overrightarrow{k}$$

Find

$$\frac{\partial^2}{\partial x \partial y}\left(\overrightarrow{A} \times \overrightarrow{B}\right) \quad \text{at} \quad (1,0,-2)$$

SOLUTION

By the definition

$$\overrightarrow{A} \times \overrightarrow{B} = \left\| \begin{array}{ccc} \overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{array} \right\| = \overrightarrow{i}\left(A_yB_z - A_zB_y\right) - \overrightarrow{j}\left(A_xB_z - A_zB_x\right) + \overrightarrow{k}\left(A_xB_y - A_yB_x\right)$$

In our case

$$\overrightarrow{A} = \underbrace{x^2yz}_{A_x}\overrightarrow{i} + \underbrace{(-2xz^3)}_{A_y}\overrightarrow{j} + \underbrace{(-xz^2)}_{A_z}\overrightarrow{k} \quad \text{and} \quad \overrightarrow{B} = \underbrace{4z}_{B_x}\overrightarrow{i} + \underbrace{y}_{B_y}\overrightarrow{j} + \underbrace{4x^2}_{B_z}\overrightarrow{k}$$

$$\overrightarrow{A}\times\overrightarrow{B} = \overrightarrow{i}\left(-2xz^3*4x^2-(-xz^2)y\right)-\overrightarrow{j}\left(x^2yz*4x^2-(-xz^2)*4z\right)+\overrightarrow{k}\left(x^2yz*y-(-2xz^3)*4z\right) =$$

$$= \overrightarrow{i}\left(xyz^2 - 8x^3z^3\right) - \overrightarrow{j}\left(4x^4yz + 4xz^3\right) + \overrightarrow{k}\left(x^2y^2z + 8xz^4\right)$$

$$\frac{\partial^2}{\partial x \partial y}\left(\overrightarrow{A} \times \overrightarrow{B}\right) = \frac{\partial^2}{\partial x \partial y}\left(\overrightarrow{i}\left(xyz^2 - 8x^3z^3\right) - \overrightarrow{j}\left(4x^4yz + 4xz^3\right) + \overrightarrow{k}\left(x^2y^2z + 8xz^4\right)\right) =$$

$$= \overrightarrow{i}\frac{\partial^2}{\partial x \partial y}\left(xyz^2 - 8x^3z^3\right) - \overrightarrow{j}\frac{\partial^2}{\partial x \partial y}\left(4x^4yz + 4xz^3\right) + \overrightarrow{k}\frac{\partial^2}{\partial x \partial y}\left(x^2y^2z + 8xz^4\right) =$$

$$= \overrightarrow{i}\frac{\partial}{\partial x}\left(xz^2\right) - \overrightarrow{j}\frac{\partial}{\partial x}\left(4x^4z\right) + \overrightarrow{k}\frac{\partial}{\partial x}\left(2x^2yz\right) = \overrightarrow{i}z^2 - \overrightarrow{j}4*4x^3z + \overrightarrow{k}2*2xyz$$

$$\frac{\partial^2}{\partial x \partial y}\left(\overrightarrow{A} \times \overrightarrow{B}\right)\Bigg|_{(1,0,-2)} = \left(\overrightarrow{i}z^2 - \overrightarrow{j}4*4x^3z + \overrightarrow{k}2*2xyz\right)\Bigg|_{(1,0,-2)} =$$

$$= \overrightarrow{i}*(-2)^2 - \overrightarrow{j}*16*1^3*(-2) + \overrightarrow{k}*4*1*0*(-2) = 4\overrightarrow{i} + 32\overrightarrow{j}$$

$$\frac{\partial^2}{\partial x \partial y}\left(\overrightarrow{A} \times \overrightarrow{B}\right)\Bigg|_{(1,0,-2)} = 4\overrightarrow{i} + 32\overrightarrow{j}$$

ANSWER

$$\frac{\partial^2}{\partial x \partial y}\left(\overrightarrow{A} \times \overrightarrow{B}\right)\Bigg|_{(1,0,-2)} = 4\overrightarrow{i} + 32\overrightarrow{j}$$

QUESTION 10

Solve

$$\frac{d^2A}{dt^2} - 4\frac{dA}{dt} - 5A = 0$$

SOLUTION

$$\frac{d^2A}{dt^2} - 4\frac{dA}{dt} - 5A = 0$$

It is ordinary, homogeneous second order differential equation with constant coefficients. The solution of this equation is found in the form (this is known from the theory of differential equations)

$$A(t) = e^{\lambda t} \Longleftrightarrow \frac{dA}{dt} = \lambda e^{\lambda t} \Longleftrightarrow \frac{d^2A}{dt^2} = \lambda^2 e^{\lambda t}$$

$$\frac{d^2A}{dt^2} - 4\frac{dA}{dt} - 5A = 0 \Longleftrightarrow \lambda^2 e^{\lambda t} - 4\lambda e^{\lambda t} - 5e^{\lambda t} = 0 \Longleftrightarrow e^{\lambda t}\big(\lambda^2 - 4\lambda - 5\big) = 0$$

$$\lambda^2 - 4\lambda - 5 = 0 \Longleftrightarrow \begin{cases} \lambda_1 = 5 \\ \lambda_2 = -1 \end{cases}$$

$$A(t) = C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t} = C_1 e^{5t} + C_2 e^{-t}$$

ANSWER

$$A(t) = C_1 e^{5t} + C_2 e^{-t}$$