Answer on Question #59205 – Math – Calculus

Question

1. Given that $A = \sin t\, i + \cos t\, j + tk$, evaluate $|\, d^2A/dt^2|$

Solution

$$A = xi + yj + zk = \sin t \cdot i + \cos t \cdot j + t \cdot k,$$
$$\frac{dA}{dt} = \frac{dx}{dt} i + \frac{dy}{dt} j + \frac{dz}{dt} k,$$

$$\frac{dA}{dt} = \frac{d(sint)}{dt} i + \frac{d(\cos t)}{dt} j + \frac{d(t)}{dt} k = \cos t \cdot i - \sin t \cdot j + k,$$

$$\frac{d^2A}{dt^2} = \frac{d^2x}{dt^2} i + \frac{d^2y}{dt^2} j + \frac{d^2z}{dt^2} k = \frac{d}{dt}\left(\frac{dx}{dt}\right) i + \frac{d}{dt}\left(\frac{dy}{dt}\right) j + \frac{d}{dt}\left(\frac{dz}{dt}\right) k,$$

$$\frac{d^2A}{dt^2} = \frac{d}{dt}(\cos t) i + \frac{d}{dt}(-\sin t) j + \frac{d}{dt}(1) k = -\sin t \cdot i - \cos t \cdot j + 0 \cdot k = -\sin t \cdot i - \cos t \cdot j.$$

Answer: $\frac{d^2A}{dt^2} = -\sin t\, i - \cos t\, j.$

Question

2. A particle moves along a curve whose parameter equations are

$$x = e^{-t}, y = 2\cos 3t \ , z = 2\sin 3t.$$

Find the magnitude of the acceleration at $t = 0$.

Solution

The velocity of a particle:

$$v(t) = \frac{dx}{dt} i + \frac{dy}{dt} j + \frac{dz}{dt} k,$$

$$v(t) = \frac{d(e^{-t})}{dt} i + \frac{d(2\cos 3t)}{dt} j + \frac{d(2\sin 3t)}{dt} k = -e^{-t} i - 6\sin 3t \cdot j + 6\cos 3t \cdot k.$$

The acceleration of a particle:

$$a(t) = \frac{d^2x}{dt^2} i + \frac{d^2y}{dt^2} j + \frac{d^2z}{dt^2} k = \frac{d}{dt}\left(\frac{dx}{dt}\right) i + \frac{d}{dt}\left(\frac{dy}{dt}\right) j + \frac{d}{dt}\left(\frac{dz}{dt}\right) k,$$

$a(t) = \frac{d}{dt}(-e^{-t})i + \frac{d}{dt}(-6\sin 3t)j + \frac{d}{dt}(6\cos 3t)k = e^{-t}i - 18\cos 3t \cdot j - 18\sin 3t \cdot k.$

So, we have the acceleration at $t = 0$:

$$a(0) = e^{-0}i - 18\cos(3\cdot 0)\cdot j - 18\sin(3\cdot 0)\cdot k = i - 18j = a_x i + a_y j + a_z k.$$

The magnitude of the acceleration at $t = 0$:

$$|a(0)| = \sqrt{a_x^2 + a_y^2 + a_z^2},$$
$$|a(0)| = \sqrt{1 + 18^2} = \sqrt{325} = 5\sqrt{13}.$$

Answer: $|a(0)| = 5\sqrt{13}.$

Question

3. A particle moves along the curve

$$x = 2t^2\,, y = t^2 - 4t,\ z = 3t - 5,$$

where t is time. Find the components of the velocity at $t = 1$ in the direction $i - 3j + 2k$

Solution

The velocity of a particle:

$$v(t) = v_x i + v_y j + v_z k = \frac{dx}{dt}i + \frac{dy}{dt}j + \frac{dz}{dt}k,$$
$$v(t) = \frac{d(2t^2)}{dt}i + \frac{d(t^2-4t)}{dt}j + \frac{d(3t-5)}{dt}k = 4ti + (2t-4)j + 3k = v_x i + v_y j + v_z k.$$

The vector projection of the velocity v onto the direction $b = b_x i + b_y j + b_z k = i - 3j + 2k$:

$$pr_b v = \frac{v\cdot b}{b\cdot b}\cdot b = \frac{v_x b_x + v_y b_y + v_z b_z}{b_x^2 + b_y^2 + b_z^2}\left(b_x i + b_y j + b_z k\right),$$
$$pr_b v = \frac{4t\cdot 1 + (2t-4)\cdot(-3) + 3\cdot 2}{1^2 + (-3)^2 + 2^2}(i - 3j + 2k),$$

$$pr_b v = \frac{4t - 6t + 12 + 6}{14}(i - 3j + 2k),$$
$$pr_b v = \frac{-t+9}{7}(i - 3j + 2k),$$
$$pr_b v(1) = \frac{-1+9}{7}(i - 3j + 2k),$$
$$pr_b v(1) = \frac{8}{7}i - \frac{24}{7}j + \frac{16}{7}k.$$

The components of the velocity at $t = 1$ in the given direction:

$$(pr_b v)_x = \frac{8}{7};\ (pr_b v)_y = -\frac{24}{7};\ (pr_b v)_z = \frac{16}{7}.$$

Answer: $(pr_b v)_x = \frac{8}{7}$; $(pr_b v)_y = -\frac{24}{7}$; $(pr_b v)_z = \frac{16}{7}$.

Question

4. Determine the unit tangent at the point where $t = 2$ on the curve

$$x = t^2 + 1\,, y = 4t - 3\,, z = 2t^2 - 6t$$

Solution

$$x'(t) = \frac{dx}{dt} = \frac{d(t^2+1)}{dt} = 2t;$$
$$y'(t) = \frac{dy}{dt} = \frac{d(4t-3)}{dt} = 4\,;$$
$$z'(t) = \frac{dz}{dt} = \frac{d(2t^2-6t)}{dt} = 4t - 6.$$

The unit tangent:

$$\tau(t) = \frac{x'(t)}{\sqrt{x'^2(t)+y'^2(t)+z'^2(t)}} i + \frac{y'(t)}{\sqrt{x'^2(t)+y'^2(t)+z'^2(t)}} j + \frac{z'(t)}{\sqrt{x'^2(t)+y'^2(t)+z'^2(t)}} k,$$

$$\tau(t) = \frac{2t}{\sqrt{4t^2+16+(4t-6)^2}} i + \frac{4}{\sqrt{4t^2+16+(4t-6)^2}} j + \frac{4t-6}{\sqrt{4t^2+16+(4t-6)^2}} k.$$

The unit tangent at the point where $t = 2$ on the curve:

$$\tau(2) = \frac{2\cdot 2}{\sqrt{4\cdot 2^2+16+(4\cdot 2-6)^2}} i + \frac{4}{\sqrt{4\cdot 2^2+16+(4\cdot 2-6)^2}} j + \frac{4\cdot 2-6}{\sqrt{4\cdot 2^2+16+(4\cdot 2-6)^2}} k,$$

$$\tau(2) = \frac{4}{\sqrt{16+16+4}} i + \frac{4}{\sqrt{16+16+4}} j + \frac{4\cdot 2-6}{\sqrt{16+16+4}} k,$$

$$\tau(2) = \frac{4}{6} i + \frac{4}{6} j + \frac{2}{6} k,$$
$$\tau(2) = \frac{2}{3} i + \frac{2}{3} j + \frac{1}{3} k,$$

Answer: $\tau(2) = \frac{2}{3} i + \frac{2}{3} j + \frac{1}{3} k.$

Question

5. If

$$A = 5t^2 i + tj - t^3 k \quad \text{and} \quad B = \sin t\, i - \cos t\, j$$

Evaluate $d/dt\,(A \cdot B)$.

Solution

If

$$A = A_x i + A_y j + A_z k = 5t^2 i + tj - t^3 k \quad \text{and}$$
$$B = B_x i + B_y j + B_z k = \sin t \cdot i - \cos t \cdot j + 0 \cdot k$$

then

$$A \cdot B = A_x B_x + A_y B_y + A_z B_z,$$

$$A \cdot B = 5t^2 \sin t + t \cdot (-\cos t) + (-t^3 \cdot 0) = 5t^2 \sin t - t \cos t.$$

$$\frac{d}{dt}(A \cdot B) = \frac{d}{dt}(5t^2 \sin t - t \cos t) =$$
$$= 10t \sin t + 5t^2 \cos t - \cos t + t \sin t = 11t \sin t + 5t^2 \cos t - \cos t.$$

Answer: $\frac{d}{dt}(A \cdot B) = 11t \sin t + 5t^2 \cos t - \cos t.$