Answer on Question #59185 – Math – Analytic Geometry

Question

Find the standard equation of the circle touching the line x + 2y = 8 at (0, 4) and passing through (3, 7).

Solution

Since the circle passes through both points A (0, 4) and B (3, 7), the distance between each point and the center of the circle O is equal to

$$AO = BO = R;$$

$$(x_0 - x_A)^2 + (y_0 - y_A)^2 = (x_0 - x_B)^2 + (y_0 - y_B)^2 = R^2;$$

$$(x_0 - 0)^2 + (y_0 - 4)^2 = (x_0 - 3)^2 + (y_0 - 7)^2;$$

$$6x_0 + 6y_0 - 42 = 0;$$

$$x_0 + y_0 - 7 = 0.$$

Therefore, the center of the circle belongs to the line
$x + y - 7 = 0.$

On the other hand, since the circle is touching the line x + 2y = 8 at A (0, 4), the radius of this circle OA is perpendicular to the line x + 2y = 8 and lies on the line b. The equation of line b can be found as follows.

The equation of the line x + 2y = 8 can be rewritten as

$$y = \frac{8 - x}{2} = -\frac{x}{2} + 4.$$

Its slope is k = - 0.5.

The slope of the line b perpendicular to the line x + 2y = 8 is

$$k_b = \frac{-1}{k} = \frac{-1}{-0.5} = 2.$$

The equation of line b:

$$y = 2x + b.$$

Since the line b passes through A (0, 4), the intercept b can be found by substitution of the coordinates of point A into the previous equation:

$$y_A = 2x_A + b.$$

$$b = y_A - 2x_A = 4 - 2 \times 0 = 4.$$

Therefore, the equation of line b is

$$y = 2x + 4.$$

Since the center of the circle belongs both to lines

$$y = 2x + 4$$

and

$$x + y - 7 = 0,$$

it can be found from the following system:

$$\begin{cases} x + y - 7 = 0 \\ y = 2x + 4 \end{cases}$$

Substituting $y = 2x + 4$ into the first equation of the system yields

$$x + 2x + 4 - 7 = 0;$$
$$3x - 3 = 0;$$
$$x = 1,$$

hence

$$y = 2x + 4 = 2 \times 1 + 4 = 6.$$

Therefore, the center of the circle is the point O(1, 6).
The radius of the circle:

$$R = \sqrt{(x_0 - x_A)^2 + (y_0 - y_A)^2} = \sqrt{(1-0)^2 + (6-4)^2} = \sqrt{5}.$$

The standard equation of the circle is

$$(x - x_0)^2 + (y - y_0)^2 = R^2;$$

$$(x-1)^2 + (y-6)^2 = (\sqrt{5})^2;$$

$$(x-1)^2 + (y-6)^2 = 5.$$

Answer:

$(x-1)^2 + (y-6)^2 = 5.$