Answer on Question #59071 – Math – Differential Equations

Question

7 Obtain the differential equation associated with the given primitive
$\ln y = Ax^2 + B$, A and B being arbitrary constants.

(I) $xy\frac{d^2y}{dx^2} - y\frac{dy}{dx} - x\left(\frac{dy}{dx}\right)^2 = 0$
(II) $xy\frac{d^2y}{dx^2} - 2y\frac{dy}{dx} = 0$
(III) $3xy\frac{d^2y}{dx^2} + 2y\frac{dy}{dx} - x\left(\frac{dy}{dx}\right)^2 = 0$
(IV) $xy\frac{d^2y}{dx^2} - x\left(\frac{dy}{dx}\right)^2 = 0$

Solution

If $\ln y = Ax^2 + B$, when $y = y(x)$, then

$$A = \frac{\ln y - B}{x^2}$$

Differentiate both sides of this equality:

$$\frac{d}{dx}(A) = \frac{d}{dx}\left(\frac{\ln y - B}{x^2}\right),$$

$$0 = \frac{x^2\frac{d}{dx}(\ln y - B) - (\ln y - B)\frac{d}{dx}(x^2)}{x^4}.$$

$$0 = \frac{x^2\frac{1}{y}\frac{dy}{dx} - 2x(\ln y - B)}{x^4}$$

or

$$x^2\frac{1}{y}\frac{dy}{dx} - 2x(\ln y - B) = 0.$$

Rewrite it as

$$x^2\frac{1}{y}\frac{dy}{dx} = 2x(\ln y - B),$$

$$\ln y - B = \frac{x}{2y}\frac{dy}{dx}.$$

Differentiate both sides of the above equality:

$$\frac{d}{dx}(\ln y - B) = \frac{d}{dx}\left(\frac{x}{2y}\frac{dy}{dx}\right).$$

$$\frac{1}{y}\frac{dy}{dx} = \frac{y\left(\frac{dy}{dx} + x\frac{d^2y}{dx^2}\right) - x\frac{dy}{dx}\cdot\frac{dy}{dx}}{2y^2},$$

$$2y\frac{dy}{dx} = y\left(\frac{dy}{dx} + x\frac{d^2y}{dx^2}\right) - x\left(\frac{dy}{dx}\right)^2,$$

or

$$\frac{xyd^2y}{dx^2} - \frac{ydy}{dx} - x\left(\frac{dy}{dx}\right)^2 = 0$$

Answer: (I) $xy\frac{d^2y}{dx^2} - y\frac{dy}{dx} - x\left(\frac{dy}{dx}\right)^2 = 0.$

Question

8 Solve $y(xy+1)dx + x(1+xy+x^2y^2)dy = 0$

(I) $3x^2y\ \ln y - 2xy - 3 = Cx^2y^2$
(II) $2x^2y^2 \ln y - 2xy - 1 = Cx^2y^2$
(III) $2x^2y^2 \ln y - xy = Cx^2y^2$
(IV) $2x^3y^2 \ln y - 2xy - 1 = Cx^2y^5$

Solution

We may use the substitution $xy = z$, $z = z(y)$. Then $\frac{dx}{dy} = x' = \frac{z'y-z}{y^2}$

$$\frac{dx}{dy} = -\frac{x(1+xy+x^2y^2)}{y(xy+1)} = -\frac{z(1+z+z^2)}{y^2(1+z)} = \frac{z'y-z}{y^2}$$

or

$$z'y = z - \frac{z(1+z+z^2)}{(1+z)} = -\frac{z^3}{1+z}.$$

We have

$$-\frac{(1+z)dz}{z^3} = \frac{dy}{y}, \qquad \int \frac{(1+z)dz}{z^3} = -\frac{1}{2z^2} - \frac{1}{z}, \qquad \int \frac{dy}{y} = \ln y.$$

So the solutions of the differential equation are

$$\frac{1}{2x^2y^2} + \frac{1}{xy} = \ln y + C$$

or

$$2x^2y^2 \ln y - 2xy - 1 = Cx^2y^2$$

Answer: (II) $2x^2y^2 \ln y - 2xy - 1 = Cx^2y^2$.