ANSWER on QUESTION #59041 – Math – Differential Equations

QUESTION #1

Solve the variable separable $x^3 dx + (y+1)^2 dy = 0$

a) $3x^4 + 4(y+2)^3 = C$
b) $5x^4 + 4(y+1)^3 = C$
c) $3x^4 + 4(y+1)^3 = C$
d) $4x^4 + 4(y+2)^3 = C$

SOLUTION

$$x^3 dx + (y+1)^2 dy = 0 \Leftrightarrow d\left(\frac{x^4}{4}\right) + d\left(\frac{(y+1)^3}{3}\right) = 0 \Leftrightarrow$$

$$d\left(\frac{x^4}{4} + \frac{(y+1)^3}{3}\right) = 0 \Leftrightarrow \frac{x^4}{4} + \frac{(y+1)^3}{3} = Const \Leftrightarrow$$

$$12 * \left(\frac{x^4}{4} + \frac{(y+1)^3}{3}\right) = \underbrace{12 * Const}_{C} \Leftrightarrow 3x^4 + 4(y+1)^3 = C$$

ANSWER: c) $3x^4 + 4(y+1)^3 = C$

QUESTION #2

Solve $(x^3 + y^3)dx - 3xy^2 dy = 0$

a) $x^5 - 2y^3 = Cx$
b) $x^3 - 2y^3 = Cx$
c) $x^3 - 3y^3 = Cx$
d) $x^5 - 2y^2 = Cx$

SOLUTION

$$(x^3 + y^3)dx - 3xy^2 dy = 0$$

$$(x^3 + y^3) - 3xy^2 \frac{dy}{dx} = 0$$

Divide both sides by x^3:

$$1+\frac{y^3}{x^3}-3\frac{xy^2}{x^3}\frac{dy}{dx}=0$$

Let

$$\frac{y^3}{x^3}=z(x)\Leftrightarrow \frac{3y^2}{x^3}\frac{dy}{dx}-\frac{3y^3}{x^4}=\frac{dz}{dx}\Leftrightarrow \frac{3y^2}{x^3}\frac{dy}{dx}=\frac{3y^3}{x^4}+\frac{dz}{dx}=\frac{3z}{x}+\frac{dz}{dx}$$

$$1+\frac{y^3}{x^3}-x\frac{3y^2}{x^3}\frac{dy}{dx}=0\Leftrightarrow 1+z-x\left(\frac{3z}{x}+\frac{dz}{dx}\right)=0\Leftrightarrow 1+z-3z-x\frac{dz}{dx}=0$$

$$1-2z=x\frac{dz}{dx}\Leftrightarrow \frac{dx}{x}=\frac{dz}{1-2z}\Leftrightarrow \ln|x|+\ln C=\frac{-1}{2}\ln|1-2z|\Leftrightarrow$$

$$-2\ln|Cx|=\ln\left|1-2\frac{y^3}{x^3}\right|\Leftrightarrow \ln|Cx|^{-2}=\ln\left|1-2\frac{y^3}{x^3}\right|\Leftrightarrow \ln\frac{1}{|Cx|^2}=\ln\left|1-2\frac{y^3}{x^3}\right|$$

$$\frac{1}{|Cx|^2}=1-2\frac{y^3}{x^3}\quad \text{multiplying by } x^3\Leftrightarrow \frac{x^3}{|Cx|^2}=x^3-2y^3\Leftrightarrow \underbrace{\frac{1}{|C|}}_{C_1}x=x^3-2y^3\Leftrightarrow$$

$$x^3-2y^3=C_1x$$

ANSWER: b) $x^3-2y^3=Cx$

QUESTION #3

Solve $\left(1+2e^{\frac{x}{y}}\right)dx+2e^{\frac{x}{y}}\left(1-\frac{x}{y}\right)dy=0$

a) $5x+2ye^{\frac{x}{y}}=C$
b) $x+2ye^{\frac{2x}{y}}=C$
c) $x+2ye^{\frac{x}{y}}=C$
d) $5x+3ye^{\frac{x}{y}}=C$

SOLUTION

We write this differential equation in the form $P(x,y)dx+Q(x,y)dy=0$, where $P(x,y)=\left(1+2e^{\frac{x}{y}}\right)$ and $Q(x,y)=2e^{\frac{x}{y}}\left(1-\frac{x}{y}\right)$. We show that this equation is an exact ordinary differential equation, To do this, check the equality

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$

$$\frac{\partial P}{\partial y} = \frac{\partial}{\partial y}\left(1 + 2e^{\frac{x}{y}}\right) = 2e^{\frac{x}{y}} * \frac{-x}{y^2}$$

$$\frac{\partial Q}{\partial x} = \frac{\partial}{\partial x}\left(2e^{\frac{x}{y}}\left(1 - \frac{x}{y}\right)\right) = 2e^{\frac{x}{y}} * \frac{1}{y}\left(1 - \frac{x}{y}\right) + 2e^{\frac{x}{y}}\left(-\frac{1}{y}\right) = 2e^{\frac{x}{y}} * \frac{1}{y}\left(1 - \frac{x}{y} - 1\right) = 2e^{\frac{x}{y}} * \frac{-x}{y^2}$$

As we can see

$$\frac{\partial P}{\partial y} = 2e^{\frac{x}{y}} * \frac{-x}{y^2} = \frac{\partial Q}{\partial x}$$

This equality confirms our guess that this ordinary differential equation is exact. Therefore,

$P(x,y) = F_x'(x,y)$ and $Q(x,y) = F_y'(x,y)$

$$F_x'(x,y) = 1 + 2e^{\frac{x}{y}} \Leftrightarrow F(x,y) = \int \left(1 + 2e^{\frac{x}{y}}\right) dx = x + 2ye^{\frac{x}{y}} + \varphi(y)$$

$$Q(x,y) = F_y'(x,y) = \frac{\partial}{\partial y}\left(x + 2ye^{\frac{x}{y}} + \varphi(y)\right) = \varphi'(y) + 2e^{\frac{x}{y}} + 2ye^{\frac{x}{y}} * \frac{-x}{y^2} =$$

$$= \varphi'(y) + 2e^{\frac{x}{y}}\left(1 - \frac{x}{y}\right) \Leftrightarrow \underbrace{2e^{\frac{x}{y}}\left(1 - \frac{x}{y}\right)}_{Q(x,y)} = \varphi'(y) + 2e^{\frac{x}{y}}\left(1 - \frac{x}{y}\right) \Leftrightarrow \varphi'(y) = 0$$

$$\varphi(y) = C_1$$

$$P(x,y)dx + Q(x,y)dy = 0 \Leftrightarrow dF(x,y) = 0 \Leftrightarrow F(x,y) = C$$

$$x + 2ye^{\frac{x}{y}} + C_1 = C \Leftrightarrow x + 2ye^{\frac{x}{y}} = \underbrace{(C - C_1)}_{C_2} \Leftrightarrow x + 2ye^{\frac{x}{y}} = C_2$$

<u>ANSWER</u>: c) $x + 2ye^{\frac{x}{y}} = C$.