Answer on **Question #:**58986, Math / Calculus

Let C[0,1] gave inner product defined by < f,g >= ∫ f(x)g(x)dx ; integrate from 0 to 1

Evaluate d(f,g) where g(x) = c (constant) and f(x) = e^x. Use calculus to find the smallest value of E(c) = d^2(f, g) and call the value of c at which this occurs g*(x) = c*. Calculate the values d(f, g*) and also d(f, g) for the case g = (1 + e)/2. Comment (one sentence).

Solution

$d(f,g)$ is distance between f,g; inner product is $\langle f,g\rangle = \int_0^1 f(x)g(x)dx$ then $E(c) = d^2(f,g) = \langle f-g, f-g\rangle = \int_0^1 \left(f(x)-g(x)\right)^2 dx = \int_0^1 (e^x - c)^2 dx = \int_0^1 (e^{2x} - 2ce^x + c^2)dx = \left(\frac{1}{2}e^{2x} - 2ce^x + c^2 x\right)\Big|_0^1 = \frac{1}{2}e^2 - 2ce + c^2 - \frac{1}{2} + 2c$. Find extrema: $\frac{dE(c)}{dc} = 2c + 2 - 2e$, $\frac{dE(c)}{dc} = 0 \Leftrightarrow 2c + 2 - 2e = 0$ and $c = e - 1 = c^* = g^*(x)$. Then $d(f,g^*) = \sqrt{E(c^*)} = \sqrt{2e - \frac{1}{2}e^2 - \frac{3}{2}} \approx 0.492$.

If $g(x) = (1+e)/2$ then $d(f,g) = \sqrt{E\left(c = \frac{1+e}{2}\right)} = \sqrt{\frac{1}{2}e - \frac{1}{4}e^2 + \frac{3}{4}} \approx 0.512$, therefore $d(f,g^*) < d(f,g)$ as expected.