Answer on Question #58981 – Math – Abstract Algebra

Question

Prove that

i) $<\mathbb{Z},+>$ is a group,

ii) $\mathbb{H} =<5\mathbb{Z},+>$ is a subgroup of $\mathbb{G} =<\mathbb{Z},+>$

Solution

i) Given any integer numbers $a \in \mathbb{Z}, b \in \mathbb{Z}$, we have to check group axioms w.r.t addition:

1. Closure:
We have $a + b = c$, where $c \in \mathbb{Z}$ is some integer. Axiom is satisfied.

2. Identity element:

We have $a + 0 = 0 + a = a$. Thus, identity element is $e = 0$. Axiom is satisfied.

3. Inverse element:

We have $a + (-a) = (-a) + a = e$. Thus, the inverse element is $a^{-1} = -a$.

Axiom is satisfied.

4. Associativity:

As $a + (b + c) = (a + b) + c$ is obviously satisfied for any integers: $a \in \mathbb{Z}, b \in \mathbb{Z}, c \in \mathbb{Z}$.

Axiom is satisfied.

Therefore, integer numbers satisfy all group axioms w.r.t addition.

ii) Now let us consider $\mathbb{H} =<5\mathbb{Z},+>$. Since $\mathbb{H} \subset \mathbb{Z}$, to prove that $\mathbb{H} =<5\mathbb{Z},+>$ is a subgroup of $<\mathbb{Z},+>$, we have to prove that $<\mathbb{H},+>$ forms a group, i.e. we must check group axioms.

1. Closure:

$$a \in \mathbb{H}, \qquad a = 5n, \ \ n = 0, \pm 1, \pm 2, \dots$$

$$b \in \mathbb{H}, \qquad a = 5m, \ \ m = 0, \pm 1, \pm 2, \dots$$

Then

$$a + b = 5n + 5m = 5(n + m) \in \mathbb{H}$$

Closure axiom is satisfied.

2. Identity element:

We have $5n + 0 = 0 + 5n = 5n$. Thus, the identity element is $e = 0 \in \mathbb{H}$. Axiom is satisfied.

3. Inverse element:

We have $5n + (-5n) = (-5n) + 5n = e$. Thus, the inverse element is $(5n)^{-1} = -5n \in \mathbb{H}$.

Axiom is satisfied.

4.Associativity:

As $5n+(5m+5k)=(5n+5m)+5k$, where $c=5k$, $k=0,\pm1,\pm2,\dots$, associativity is obviously satisfied for any integers $n\in\mathbb{Z}, m\in\mathbb{Z}, k\in\mathbb{Z}$. Axiom is satisfied.

Therefore, we proved that $\mathbb{H}=<5\mathbb{Z},+>$ is a subgroup of $\mathbb{G}=<\mathbb{Z},+>$.

Answer: $<\mathbb{Z},+>$ is a group, $\mathbb{H}=<5\mathbb{Z},+>$ is a subgroup of $\mathbb{G}=<\mathbb{Z},+>$.