Answer on Question #58980 – Math – Abstract Algebra

Question

Do the non-zero positive rational numbers form a group with respect to multiplication?

Solution

Let $G = \{x: x > 0, x \in \mathbb{Q}\}$.

Let us check the group axioms:

Closure. Let $a, b \in G$. Then $a \cdot b > 0$, and obviously $a \cdot b \in \mathbb{Q}$.

Associativity. Obviously $(a \cdot b) \cdot c = a \cdot (b \cdot c) = abc$ for all $a, b, c \in G$.

Identity element. There exists an element $e \coloneqq 1 \in G$: $1 \cdot a = a \cdot 1 = a$ for all $a \in G$.

Inverse element. For each $a \in G$ there exists an element $a^{-1} \coloneqq \frac{1}{a} \in G$ $\left(\frac{1}{a} > 0, \frac{1}{a} \in \mathbb{Q}\right)$ such that

$a \cdot \frac{1}{a} = \frac{1}{a} \cdot a = 1 = e$.

All the axioms are satisfied so the non-zero positive rational numbers form a group with respect to multiplication.

Answer. Yes.

Question

Do the even integer form a group with respect to addition?

Solution

Let $G = \{x: x \in \mathbb{Z},\ 2|x\}$. Let us check the group axioms:

Closure. Let $a = 2k_1 \in G, b = 2k_2 \in G$. Then $a + b = 2k_1 + 2k_2 = 2(k_1 + k_2) \in G$.

Associativity. Obviously $(a + b) + c = a + (b + c) = a + b + c$ for all $a, b, c \in G$.

Identity element. There exists an element $e \coloneqq 0 \in G$: $0 + a = a + 0 = a$ for all $a \in G$.

Inverse element. For each $a \in G$ there exists an element $a^{-1} \coloneqq -a \in G$ $\left(-a \in \mathbb{Z}, 2|(-a)\right)$ such that

$a + (-a) = -a + a = 0 = e$.

All the axioms are satisfied so the even integer form a group with respect to addition.

Answer. Yes.