Answer on Question #58902 – Math – Calculus

Question

Find $f(x, y)$ such that $z = f(x, y)$ defines a plane and

$$\iint f(x, y)dA = \iint xf(x, y)dA = 0, \quad f(1,2) = -1$$

where $\mathfrak{D}$ is the region bounded by the graphs of $x + y = 3, x = 0,$ and $y = 0$.

Solution

Since $f(x, y)$ defines a plane we have

$$z = f(x, y) = ax + by + c,$$

where $a, b,$ and c are some unknown constants.

As $f(1,2) = -1$ we obtain the first equation:

$$a + 2b + c = -1 .$$

To find two other equations we have to calculate two integrals:

$$I_1 = \iint f(x, y)dA = \int_0^3 dx \int_0^{3-x} dy(ax + by + c) = 0$$

and

$$I_2 = \iint xf(x, y)dA = \int_0^3 dx \int_0^{3-x} dy\, x(ax + by + c) = 0$$

because $\mathfrak{D}$ is bounded by axes and by $y = 3 - x$. Thus, $0 \le x \le 3, 0 \le y \le 3 - x$.

We have

$$I_1 = \int_0^3 dx \int_0^{3-x} dy(ax + by + c) = \int_0^3 dx \left\{(3 - x)(ax + c) + \frac{b}{2}(3 - x)^2\right\}$$

As $I_1 = 0$ we get

$$a + b + c = 0.$$

$$I_2 = \int_0^3 dx \int_0^{3-x} dy\, x(ax + by + c) = \int_0^3 dx \left\{x(3 - x)(ax + c) + x\frac{b}{2}(3 - x)^2\right\}$$

As $I_2 = 0$ we get:

$$6a + 3b + 4c = 0.$$

Finally we obtain the system of equations:

$$\begin{cases} a + 2b + c = -1, \\ a + b + c = 0, \\ 6a + 3b + 4c = 0. \end{cases}$$

Therefore, $a = -\frac{1}{2}$, $b = -1, c = \frac{3}{2}$.

The function is given by

$$f(x, y) = -\frac{x}{2} - y + \frac{3}{2}.$$

Answer: $f(x, y) = -\frac{x}{2} - y + \frac{3}{2}$.