Answer on Question #58883 – Math – Differential Equations

Question

5. Derive the differential equation for the area bounded by the arc of a curve, the x- axis, and the two ordinates, one fixed and one variable, is equal to thrice the length of the arc between the ordinates.

Solution

The equation of area:

$$S = \int_{x_0}^{x} y(x)dx,$$

where x_0 is a fixed ordinate, x is variable ordinate.

The equation of arc length:

$$L = \int_{x_0}^{x} \sqrt{1 + (y'(x))^2}\, dx$$

It is given that $S = 3L$, which is equivalent to

$$\int_{x_0}^{x} y(x)dx = 3\int_{x_0}^{x} \sqrt{1 + (y'(x))^2}\, dx \qquad (1)$$

Differentiate both sides of (1) with respect to x:

$$y(x) = 3\sqrt{1 + \left(y'(x)\right)^2} \qquad (2)$$

Differentiating both sides of (2) with respect to x derive the differential equation:

$$y' = \frac{6y'y''}{2\sqrt{1 + \left(y'(x)\right)^2}}$$

$$y' = \frac{3y'y''}{\sqrt{1+\left(y'(x)\right)^2}} \qquad (3)$$

If $y' = 0$, then $y = C$ is a straight line, where C is an arbitrary constant;

$$S = \int_{x_0}^{x} y(x)dx = \int_{x_0}^{x} Cdx = C(x - x_0),$$

$L = \int_{x_0}^{x} \sqrt{1 + (y'(x))^2}\, dx = \int_{x_0}^{x} \sqrt{1 + (C')^2}\, dx = \int_{x_0}^{x} dx = (x - x_0).$

If $S = 3L$, then $C(x - x_0) = 3(x - x_0)$, hence $C = 3$ and $y = 3$, its differential equation is $y' = 0$.

If $y' \neq 0$, then we do not deal with a straight line, $y = y(x)$ represents a curve.

Divide both sides of (3) by y':

$$1 = \frac{3y''}{\sqrt{1 + \left(y'(x)\right)^2}}$$

$$\sqrt{1 + (y'(x))^2} = 3y''$$

Square both sides:

$$(y')^2 + 1 = 9(y'')^2$$

Answer: $(y')^2 + 1 = 9(y'')^2$

Question

6. Find the differential equation of all straight lines at a unit distance from the origin.

Solution

The equation of a circle centered in the origin and the radius of 1:

$$x^2 + y^2 = 1$$

Differentiating both sides with respect to x derive the differential equation:

$$2yy' = -2x;$$

$y' = -\frac{x}{y}$ is the slope of the lines at distance 1 from the origin.

From $x^2 + y^2 = 1$ it follows that $y = \sqrt{1 - x^2}$ and using $y' = -\frac{x}{y}$ we come to

$$y' = -\frac{x}{\sqrt{1-x^2}}.$$

Answer: $y' = -\frac{x}{\sqrt{1-x^2}}$.

Question

7. Obtain the differential equation associated with the given primitive

$\ln y = Ax^2 + B$

A and *B* being arbitrary constant.

Solution

Differentiate both sides of $\ln y = Ax^2 + B$ with respect to x and derive the equation:

$$\frac{y'}{y} = 2Ax$$

$$\frac{y'}{xy} = 2A$$

Take the logarithm:

$$\ln y' - \ln x - \ln y = \ln 2A$$

Differentiate both sides of the previous formula with respect to x:

$$\frac{y''}{y'} - \frac{1}{x} - \frac{y'}{y} = 0.$$

Answer: $\frac{y''}{y'} - \frac{1}{x} - \frac{y'}{y} = 0.$