Answer on Question #58826 – Math – Calculus

Question

Integration of ((sin x)^2 + 4)/((sin x)^4 +16)dx

Solution

Let's recall some trigonometrical formulas:

$\frac{1}{\cos^2 x} = 1 + tan^2 x; \cos^2 x = \frac{1}{1+tan^2 x}; (tanx)' = \frac{1}{\cos^2 x}.$

Next,

$$\int \frac{\sin^2 x + 4}{\sin^4 x + 16} dx = \int \frac{\frac{\sin^2 x}{\cos^2 x} + \frac{4}{\cos^2 x}}{\frac{(sin^2 x)^2}{\cos^2 x} + \frac{16}{\cos^2 x}} dx = \int \frac{tan^2 x + 4(1 + tan^2 x)}{tan^2 x(1 - \cos^2 x) + 16(1 + tan^2 x)} dx =$$

$$= \int \frac{5tan^2 x + 4}{tan^2 x\left(1 - \frac{1}{1 + tan^2 x}\right) + 16(1 + tan^2 x)} dx$$

$$= \int \frac{5tan^2 x + 4}{tan^2 x\left(\frac{tan^2 x}{1 + tan^2 x}\right) + 16(1 + tan^2 x)} dx =$$

$$= \left[tanx = t; x = arctant; dx = \frac{1}{1 + t^2} dt\right] = \int \frac{5t^2 + 4}{(1 + t^2)\left(\frac{t^4}{1 + t^2}\right) + 16(1 + t^2)^2} dt =$$

$$= \int \frac{5t^2 + 4}{t^4 + 16 + 32t^2 + 16t^4} dt = \int \frac{5t^2 + 4}{17t^4 + 32t^2 + 16} dt = [D = 32^2 - 4 * 17 * 16 = -64;$$

$$t^2 = \frac{-32 + 8i}{34}, t^2 = \frac{-32 - 8i}{34}] = \int \frac{5t^2 + 4}{17(t^2 + \frac{16 - 4i}{17})(t^2 + \frac{16 + 4i}{17})} dt =$$

$$[\frac{A}{\left(t^2 + \frac{16 - 4i}{17}\right)} + \frac{B}{\left(t^2 + \frac{16 + 4i}{17}\right)} = \frac{5t^2 + 4}{(t^2 + \frac{16 - 4i}{17})(t^2 + \frac{16 + 4i}{17})}$$

$$A\left(t^2 + \frac{16 + 4i}{17}\right) + B\left(t^2 + \frac{16 - 4i}{17}\right) =$$

$$= 5t^2 + 4; \begin{cases} A + B = 5 \\ A\frac{16 + 4i}{17} + B\frac{16 - 4i}{17} = 4 \end{cases} \begin{cases} B = 5 - A \\ A\frac{16 + 4i}{17} + (5 - A)\frac{16 - 4i}{17} = 4 \end{cases}$$

$$\begin{cases} B = 5 - A \\ A(16 + 4i - 16 + 4i) + 80 - 20i = 68 \end{cases} \begin{cases} B = \frac{5}{2} - \frac{3}{2}i \\ A = \frac{-12 + 20i}{8i} = \frac{5}{2} + \frac{3}{2}i \end{cases}]$$

$$=\frac{1}{17}*\frac{1}{2}\int\left(\frac{5+3i}{\left(t^2+\frac{16-4i}{17}\right)}+\frac{5-3i}{\left(t^2+\frac{16+4i}{17}\right)}\right)dt=$$

$$=\frac{1}{34}\left(\frac{(5+3i)\sqrt{17}}{2\sqrt{4-i}}arctan\left(\frac{t\sqrt{17}}{2\sqrt{4-i}}\right)+\frac{(5-3i)\sqrt{17}}{2\sqrt{4+i}}arctan\left(\frac{t\sqrt{17}}{2\sqrt{4+i}}\right)\right)+C$$

$$=$$

$$=\frac{1}{68}\left(\frac{(5+3i)\sqrt{17}}{\sqrt{4-i}}*\frac{\sqrt{4+i}}{\sqrt{4+i}}arctan\left(\frac{t\sqrt{17}}{2\sqrt{4-i}}*\frac{\sqrt{4+i}}{\sqrt{4+i}}\right)+\frac{(5-3i)\sqrt{17}}{\sqrt{4+i}}*\right.$$

$$\left.*\frac{\sqrt{4-i}}{\sqrt{4-i}}arctan\left(\frac{t\sqrt{17}}{2\sqrt{4+i}}*\frac{\sqrt{4-i}}{\sqrt{4-i}}\right)\right)+C$$

$$=\left[\sqrt{(4-i)(4+i)}=\sqrt{17};t=tanx\right]$$

$$=\frac{1}{68}\left((5+3i)\sqrt{4+i}*arctan\left(\frac{tanx*\sqrt{4+i}}{2}\right)+(5-3i)\sqrt{4-i}\right.$$

$$\left.*arctan\left(\frac{tanx*\sqrt{4-i}}{2}\right)\right)+C,$$

where C is an integration constant.

Answer: $\int\frac{\sin^2x+4}{\sin^4x+16}dx=\frac{1}{68}\left((5+3i)\sqrt{4+i}*arctan\left(\frac{tanx*\sqrt{4+i}}{2}\right)+\right.$

$\left.+(5-3i)\sqrt{4-i}*arctan\left(\frac{tanx*\sqrt{4-i}}{2}\right)\right)+C.$