Answer on Question #58821 – Math – Differential Equations

Question

1. Solve $(x^3 + y^3)dx - 3xy^2dy = 0$

Solution

$(x^3 + y^3)dx - 3xy^2dy = 0;$

$(x^3 + y^3)dx = 3xy^2dy;$

$3xy^2dy = (x^3 + y^3)dx;$

Divide both sides by $3xy^2dx$:

$\frac{dy}{dx} = \frac{x^3+y^3}{3xy^2}.$

Divide numerator and denominator by x^3.

$\frac{dy}{dx} = \frac{\frac{x^3}{x^3}+\frac{y^3}{x^3}}{\frac{3xy^2}{x^3}};$

$\frac{dy}{dx} = \frac{1+\frac{y^3}{x^3}}{3\frac{y^2}{x^2}};$

Let $u = \frac{y}{x}$, then $y = ux$, $\frac{dy}{dx} = y' = u'x + u$.

$u'x + u = \frac{1+u^3}{3u^2};$

$u'x = \frac{1+u^3}{3u^2} - u;$

$u'x = \frac{1+u^3-3u^3}{3u^2};$

$u'x = \frac{1-2u^3}{3u^2};$

$\frac{du}{dx}x = \frac{1-2u^3}{3u^2};$

Separate the variables:

$\frac{3u^2du}{1-2u^3} = \frac{dx}{x};$

Integrate both sides:

$\int \frac{3u^2du}{1-2u^3} = \int \frac{dx}{x};$

$-\frac{1}{2}\int \frac{d(1-2u^3)}{1-2u^3} = \int \frac{dx}{x};$

$-\frac{1}{2}\ln|1-2u^3| = \ln x + \ln C$, where C is an integration constant;

$\ln\frac{1}{\sqrt{1-2u^3}} = \ln|Cx|$.

$\frac{1}{\sqrt{1-2u^3}} = Cx;$

$1-2u^3 = \frac{1}{Cx^2};$

$2u^3 = 1-\frac{1}{Cx^2};$

$2\frac{y^3}{x^3} = 1-\frac{1}{Cx^2};$

$y^3 = \frac{x^3}{2}-\frac{x^3}{2Cx^2};$

$y^3 = \frac{x^3}{2}-\frac{x}{2C}.$

Then $y = \sqrt[3]{\frac{x^3}{2}-\frac{x}{2C}}.$

Answer: $y = \sqrt[3]{\frac{x^3}{2}-\frac{x}{2C}}.$

Question

2. Solve the variable separable $x^3dx+(y+1)^2dy = 0.$

Solution

$x^3dx+(y+1)^2dy = 0;$

$x^3dx = -(y+1)^2dy.$

Separate the variables:

$x^3dx = -(y+1)^2dy.$

Integrate both sides:

$\int x^3dx = -\int (y+1)^2\, dy;$

$\frac{x^4}{4}+C = -\int (y+1)^2dy$, where C is an integration constant;

$-\int (y+1)^2 d(y+1) = \frac{x^4}{4}+C;$

$-\frac{(y+1)^3}{3} = \frac{x^4}{4}+C;$

$(y+1)^3 = -\frac{3x^4}{4}-3C;$

$y+1=\sqrt[3]{-\frac{3x^4}{4}-3C};$

$y=-1+\sqrt[3]{-\frac{3x^4}{4}-3C}.$

Answer: $y=-1+\sqrt[3]{-\frac{3x^4}{4}-3C}.$

Question

3. Solve $\left(1+2e^{\frac{x}{y}}\right)dx+2e^{\frac{x}{y}}\left(1-\frac{x}{y}\right)dy=0.$

Solution

$\left(1+2e^{\frac{x}{y}}\right)dx+2e^{\frac{x}{y}}\left(1-\frac{x}{y}\right)dy=0.$

Let $P(x,y)=1+2e^{\frac{x}{y}}$, $Q(x,y)=2e^{\frac{x}{y}}\left(1-\frac{x}{y}\right)=2e^{\frac{x}{y}}-2e^{\frac{x}{y}}\frac{x}{y}$.

Find $\frac{\partial P(x,y)}{\partial y}=-\frac{2xe^{\frac{x}{y}}}{y^2}$ and $\frac{\partial Q(x,y)}{\partial x}=-\frac{2xe^{\frac{x}{y}}}{y^2}$. Because $\frac{\partial P(x,y)}{\partial y}=\frac{\partial Q(x,y)}{\partial x}$, we deal with an exact equation.

Define $f(x,y)$ such that $\frac{\partial f(x,y)}{\partial x}=P(x,y)$ and $\frac{\partial f(x,y)}{\partial y}=Q(x,y)$.

The solution is $f(x,y)=C$, where C is an arbitrary real constant;

$f(x,y)=\int P(x,y)dx+g(y)=\int\left(1+2e^{\frac{x}{y}}\right)dx+g(y)=x+2ye^{\frac{x}{y}}+g(y).$

$f(x,y)=\int Q(x,y)dy+r(x)=\int 2e^{\frac{x}{y}}\left(1-\frac{x}{y}\right)dy+r(x)=2ye^{\frac{x}{y}}+r(x).$

Equate both expressions for $f(x,y)$:

$x+2ye^{\frac{x}{y}}+g(y)=2ye^{\frac{x}{y}}+r(x),$

hence $g(y)=0$ and $r(x)=x$, therefore $f(x,y)=x+2ye^{\frac{x}{y}}=C$

Answer: $x+2ye^{\frac{x}{y}}=C.$

Question

4. Solve $y(xy+1)dx + x(1+xy+x^2y^2)dy = 0$

Solution

Divide both sides by dx:

$y(xy+1) + x(1+xy+x^2y^2)\frac{dy}{dx} = 0;$

Let $u = xy$. Then $y = \frac{u}{x}$, $\frac{dy}{dx} = y' = \frac{u'}{x} - \frac{u}{x^2}$;

$\frac{u}{x}(u+1) + x(1+u+u^2)\left(\frac{u'}{x} - \frac{u}{x^2}\right) = 0;$

Multiply both sides by x:

$u^2 + u + (1+u+u^2)(u'x - u) = 0;$

$u^2 + u + u'x(1+u+u^2) - u - u^2 - u^3 = 0;$

$u'x(1+u+u^2) - u^3 = 0;$

$u'x(1+u+u^2) = u^3;$

$\frac{du}{dx}x(1+u+u^2) = u^3.$

Separate the variables:

$\frac{(1+u+u^2)du}{u^3} = \frac{dx}{x}.$

Integrate both sides:

$\int \frac{(1+u+u^2)du}{u^3} = \int \frac{dx}{x};$

$\int u^{-3}du + \int u^{-2}du + \int \frac{du}{u} = \ln x + C;$

C is an integration constant;

$\frac{1}{-2u^2} - \frac{1}{u} + \ln u = \ln x + C.$

Go back to $u = xy$, then find $\frac{1}{-2x^2y^2} - \frac{1}{xy} + \ln(xy) = \ln x + C;$

$\frac{1}{-2x^2y^2} - \frac{1}{xy} + \ln x + \ln y = \ln x + C;$

$\frac{1}{-2x^2y^2} - \frac{1}{xy} + \ln y = C.$

Answer: $-\frac{1}{2x^2y^2} - \frac{1}{xy} + \ln y = C.$

Question

5. Solve $xdy - ydx - \sqrt{x^2 - y^2}dx = 0$

Solution

$xdy - ydx - \sqrt{x^2 - y^2}dx = 0;$

$xdy = \left(y + \sqrt{x^2 - y^2}\right)dx.$

Divide both sides by xdx:

$\frac{dy}{dx} = \frac{\left(y+\sqrt{x^2-y^2}\right)}{x};$

$\frac{dy}{dx} = \frac{y}{x} + \sqrt{1 - \frac{y^2}{x^2}};$

Let $u = \frac{y}{x}$, then $y = ux$, $\frac{dy}{dx} = y' = u'x + u.$

$u'x + u = u + \sqrt{1 - u^2};$

$u'x = \sqrt{1 - u^2};$

$\frac{du}{dx}x = \sqrt{1 - u^2}.$

Separate the variables:

$\frac{du}{\sqrt{1-u^2}} = \frac{dx}{x}.$

Integrate both sides:

$\int \frac{du}{\sqrt{1-u^2}} = \int \frac{dx}{x};$

$\sin^{-1} u = \ln x + \ln C;$

C is an integration constant.

$u = \sin(\ln Cx).$

Go back to $u = \frac{y}{x}$, then $\frac{y}{x} = \sin(\ln Cx),$

$y = x\sin(\ln Cx).$

Answer: $y = x\sin(\ln Cx).$