Answer on Question #58765 – Math – Linear Algebra

Question

Solve the set of linear equations by the matrix method:

$$\begin{cases} a + 3b + 2c = 3 \\ 2a - b - 3c = -8 \\ 5a + 2b + c = 9 \end{cases}$$

Solution

$Ax = y,$

$A^{-1}Ax = A^{-1}y,$

$x = A^{-1}y,$

$$A = \begin{pmatrix} 1 & 3 & 2 \\ 2 & -1 & -3 \\ 5 & 2 & 1 \end{pmatrix}, \quad x = \begin{pmatrix} a \\ b \\ c \end{pmatrix}, \quad y = \begin{pmatrix} 3 \\ -8 \\ 9 \end{pmatrix},$$

$$\begin{pmatrix} 1 & 3 & 2 \\ 2 & -1 & -3 \\ 5 & 2 & 1 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 3 \\ -8 \\ 9 \end{pmatrix}.$$

The inverse of a matrix A is

$$A^{-1} = \frac{1}{|A|} A_*^T.$$

The determinant of a matrix A is

$$|A| = \begin{vmatrix} 1 & 3 & 2 \\ 2 & -1 & -3 \\ 5 & 2 & 1 \end{vmatrix} = 1 \cdot \begin{vmatrix} -1 & -3 \\ 2 & 1 \end{vmatrix} - 3 \begin{vmatrix} 2 & -3 \\ 5 & 1 \end{vmatrix} + 2 \begin{vmatrix} 2 & -1 \\ 5 & 2 \end{vmatrix} =$$
$$= 1 \cdot \big(-1 \cdot 1 - 2 \cdot (-3)\big) - 3 \cdot \big(2 \cdot 1 - 5 \cdot (-3)\big) + 2 \cdot \big(2 \cdot 2 - 5 \cdot (-1)\big) =$$
$$= -1 + 6 - 3(2 + 15) + 2(4 + 5) = 5 - 51 + 18 = -28.$$

The minor matrix is

$$M = \begin{pmatrix} M_{11} & M_{12} & M_{13} \\ M_{21} & M_{22} & M_{23} \\ M_{31} & M_{32} & M_{33} \end{pmatrix}.$$

The cofactor matrix is

$$A_* = \begin{pmatrix} M_{11} & -M_{12} & M_{13} \\ -M_{21} & M_{22} & -M_{23} \\ M_{31} & -M_{32} & M_{33} \end{pmatrix}.$$

Minors are

$$M_{11} = \begin{vmatrix} -1 & -3 \\ 2 & 1 \end{vmatrix} = -1 \cdot 1 - 2 \cdot (-3) = -1 + 6 = 5,$$

$$M_{12} = \begin{vmatrix} 2 & -3 \\ 5 & 1 \end{vmatrix} = 2 \cdot 1 - 5 \cdot (-3) = 2 + 15 = 17,$$

$$M_{13} = \begin{vmatrix} 2 & -1 \\ 5 & 2 \end{vmatrix} = 2 \cdot 2 - 5 \cdot (-1) = 9,$$

$$M_{21} = \begin{vmatrix} 3 & 2 \\ 2 & 1 \end{vmatrix} = 3 \cdot 1 - 2 \cdot 2 = -1,$$

$$M_{22} = \begin{vmatrix} 1 & 2 \\ 5 & 1 \end{vmatrix} = 1 \cdot 1 - 5 \cdot 2 = -9,$$

$$M_{23} = \begin{vmatrix} 1 & 3 \\ 5 & 2 \end{vmatrix} = 1 \cdot 2 - 5 \cdot 3 = -13,$$

$$M_{31} = \begin{vmatrix} 3 & 2 \\ -1 & -3 \end{vmatrix} = 3 \cdot (-3) - (-1) \cdot 2 = -7,$$

$$M_{32} = \begin{vmatrix} 1 & 2 \\ 2 & -3 \end{vmatrix} = 1 \cdot (-3) - 2 \cdot 2 = -7,$$

$$M_{33} = \begin{vmatrix} 1 & 3 \\ 2 & -1 \end{vmatrix} = 1 \cdot (-1) - 2 \cdot 3 = -7.$$

The cofactor matrix will be

$$A_* = \begin{pmatrix} 5 & -17 & 9 \\ 1 & -9 & 13 \\ -7 & 7 & -7 \end{pmatrix}.$$

The transpose of matrix A_* is

$$A_*^T = \begin{pmatrix} 5 & 1 & -7 \\ -17 & -9 & 7 \\ 9 & 13 & -7 \end{pmatrix}.$$

The inverse of a matrix A is

$$A^{-1} = -\frac{1}{28}\cdot\begin{pmatrix} 5 & 1 & -7 \\ -17 & -9 & 7 \\ 9 & 13 & -7 \end{pmatrix} = \begin{pmatrix} -\frac{5}{28} & -\frac{1}{28} & \frac{1}{4} \\ \frac{17}{28} & \frac{9}{28} & -\frac{1}{4} \\ -\frac{9}{28} & -\frac{13}{28} & \frac{1}{4} \end{pmatrix}.$$

The solution of the system is

$$x = \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} -\frac{5}{28} & -\frac{1}{28} & \frac{1}{4} \\ \frac{17}{28} & \frac{9}{28} & -\frac{1}{4} \\ -\frac{9}{28} & -\frac{13}{28} & \frac{1}{4} \end{pmatrix}\begin{pmatrix} 3 \\ -8 \\ 9 \end{pmatrix} = \begin{pmatrix} \frac{-5\cdot 3}{28} + \frac{-1\cdot(-8)}{28} + \frac{1\cdot 9}{4} \\ \frac{17\cdot 3}{28} + \frac{9\cdot(-8)}{28} + \frac{(-1)\cdot 9}{4} \\ \frac{-9\cdot 3}{28} + \frac{-13\cdot(-8)}{28} + \frac{1\cdot 9}{28} \end{pmatrix}$$

$$= \begin{pmatrix} -\frac{15}{28} + \frac{8}{28} + \frac{9}{4} \\ \frac{51}{28} - \frac{72}{28} - \frac{9}{4} \\ -\frac{27}{28} + \frac{104}{28} + \frac{9}{4} \end{pmatrix} =$$

$$= \begin{pmatrix} \frac{-15+8+9\cdot 7}{28} \\ \frac{51-72-9\cdot 7}{28} \\ \frac{-27+104+7\cdot 9}{28} \end{pmatrix} = \begin{pmatrix} \frac{56}{28} \\ \frac{-84}{28} \\ \frac{140}{28} \end{pmatrix} = \begin{pmatrix} 2 \\ -3 \\ 5 \end{pmatrix},$$

that is, $a = 2$, $b = -3$, $c = 5$.

Answer: $a = 2$; $b = -3$; $c = 5$.