Answer on Question #58697 – Math – Linear Algebra

Question

(a) Let B = {a1, a2, a3} be an ordered basis of R3 with a1 = (1, 0, -1), a2 = (1, 1, 1), a3 = (1, 0, 0). Write the vector v = (a, b, c) as a linear combination of the basis vectors from B.

Solution

We note that the vectors {***a1***, ***a2***, ***a3***} are linearly independent. If ***v*** = (a, b, c) is the linear combination of {***a1***, ***a2***, ***a3***}, then $\boldsymbol{v} = \begin{bmatrix} a \\ b \\ c \end{bmatrix} = x_1\boldsymbol{a1} + x_2\boldsymbol{a2} + x_3\boldsymbol{a3} = x_1\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + x_2\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + x_3\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, where unknown constants are $x_1, x_2, x_3\, \epsilon R$.

Answer: $\begin{bmatrix} a \\ b \\ c \end{bmatrix} = x_1\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + x_2\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + x_3\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$.

Question

(b) Suppose a1 = (1, 0, 1), a2 = (0, 1, -2) and a3 = (-1, -1, 0) are vectors in R3 and f: R3 -> R is a linear functional such that f(a1) = 1, f(a2) = -1 and f(a3) = 3. If a= (a, b, c) ϵ R3, find f(a).

Solution

We note that the linear functional has properties: $f(\boldsymbol{u}+\boldsymbol{v}) = f(\boldsymbol{u}) + f(\boldsymbol{v})$ and $f(c\boldsymbol{u}) = c\, f(\boldsymbol{u})$, where c is a constant. It is known that the inner product has these properties. Let the unknown vector $\boldsymbol{w}$ be $\boldsymbol{w}(w_1, w_2, w_3)$ and the dependent linear functional is $f(\boldsymbol{u}) = \boldsymbol{u}\cdot\boldsymbol{w}$. We have conditions:

$$f(\boldsymbol{a1}) = 1 \Leftrightarrow \boldsymbol{a1}\cdot\boldsymbol{w} = 1 \Leftrightarrow (1,0,1)\cdot(w_1, w_2, w_3) = 1 \Leftrightarrow 1w_1 + 0 + 1w_3 = 1;$$

$$f(\boldsymbol{a2}) = -1 \Leftrightarrow \boldsymbol{a2}\cdot\boldsymbol{w} = -1 \Leftrightarrow (0,1,-2)\cdot(w_1, w_2, w_3) = -1 \Leftrightarrow 0 + 1w_2 - 2w_3 = -1;$$

$$f(\boldsymbol{a3}) = 3 \Leftrightarrow \boldsymbol{a3}\cdot\boldsymbol{w} = 3 \Leftrightarrow (-1,-1,0)\cdot(w_1, w_2, w_3) = 3 \Leftrightarrow -1w_1 - 1w_2 + 0 = 3.$$

We have the system of linear equations:

$$\begin{cases} 1w_1 + 0 + 1w_3 = 1, \\ 0 + 1w_2 - 2w_3 = -1, \\ -1w_1 - 1w_2 + 0 = 3. \end{cases}$$

that gives the solution

$$\begin{cases} w_1 = 4, \\ w_2 = -7, \\ w_3 = -3. \end{cases}$$

Then

$\boldsymbol{w}(w_1, w_2, w_3) = (4, -7, -3)$ and

$f(\boldsymbol{a}) = \boldsymbol{a}\cdot\boldsymbol{w} = (a, b, c)(4, -7, -3) = 4a - 7b - 3c.$

Answer: $f(\boldsymbol{a}) = 4a - 7b - 3c.$