ANSWER on Question #58667 – Math – Differential Equations

QUESTION

(25 marks) Find $\alpha > 0$ such that $y_1(x) = x^\alpha$ is a solution of the following differential equation:

$$y'' - \frac{y'}{x} + \frac{y}{x^2} = 0, \qquad x \in (0; +\infty). \tag{1}$$

Solve the given differential equation.

SOLUTION

If $y_1(x) = x^\alpha$ is a solution of the differential equation (1), then

$$y_1'' - \frac{y_1'}{x} + \frac{y_1}{x^2} = 0,$$

$$y_1'' - \frac{y_1'}{x} + \frac{y_1}{x^2} = \alpha(\alpha-1)x^{\alpha-2} - \frac{\alpha x^{\alpha-1}}{x} + \frac{x^\alpha}{x^2}$$
$$= \alpha(\alpha-1)x^{\alpha-2} - \alpha x^{\alpha-2} + x^{\alpha-2} = 0$$

$$(\alpha(\alpha-1) - \alpha + 1)x^{\alpha-2} = 0 \ \rightarrow \alpha(\alpha-1) - \alpha + 1 = 0$$

$$\alpha(\alpha-1) - \alpha + 1 = \alpha^2 - \alpha - \alpha + 1 = \alpha^2 - 2\alpha + 1$$
$$= (\alpha-1)^2 = 0 \rightarrow \ \alpha = 1$$

$y_1(x) = x$ is the first solution of the differential equation (1).

It follows from Liouville's formula that the second solution is given by

$$y_2(x) = y_1 \int \frac{e^{-\int p(x)dx}}{y_1^2} dx = x \int \frac{e^{-\int \frac{-1}{x}dx}}{x^2} dx = x \int \frac{e^{\ln(x)}}{x^2} dx = x \int \frac{x}{x^2} dx =$$

$$= x \int \frac{1}{x} dx = xln(x)$$

The general solution of the differential equation (1) is

$$y(x) = C_1 y_1(x) + C_2 y_2(x) = C_1 x + C_2 xln(x)$$

Answer: $\alpha = 1$; $y(x) = C_1 x + C_2 x ln(x)$.