Answer on Question #58666 – Math – Differential Equations

Question

$y'' - 3y' = 8e^{3x} + 4sinx.$

Solution

The general solution will be the sum of the complementary solution y_0 and a particular solution y^*:

$$y = y_0 + y^*.$$

Solving the characteristic equation

$$k^2 - 3k = 0$$

gives

$$k_1 = 0,\ k_2 = 3.$$

Then the solution y_0 of the differential equation $y'' - 3y' = 0$ is

$$y_0 = C_1 + C_2e^{3x}.$$

Find a particular solution y^* of the differential equation

$$y'' - 3y' = 8e^{3x} + 4sinx$$

by the method of undetermined coefficients:

$$y^* = xAe^{3x} + Bcosx + Csinx.$$

Solve for unknown constants A, B, C:

$y^{*\prime} = Ae^{3x} + 3Axe^{3x} - Bsinx + Ccosx;$

$y^{*\prime\prime} = 3Ae^{3x} + 3Ae^{3x} + 9xAe^{3x} - Bcosx - Csinx = 6Ae^{3x} + 9xAe^{3x} - Bcosx - Csinx.$

Plug a particular solution into the differential equation:

$6Ae^{3x} + 9xAe^{3x} - Bcosx - Csinx - 3(Ae^{3x} + 3Axe^{3x} - Bsinx + Ccosx) = 8e^{3x} + 4sinx.$

$3Ae^{3x} + (-B - 3C)cosx + (3B - C)sinx = 8e^{3x} + 4sinx.$

Equating like terms gives the system of equations:

$$\begin{cases} 3A = 8, \\ -B - 3C = 0, \\ 3B - C = 4. \end{cases} \Rightarrow \begin{cases} A = \frac{8}{3} \\ B = -3C \\ -9C - C = 4 \end{cases} \Rightarrow \begin{cases} A = \frac{8}{3} \\ B = -3 \cdot \left(-\frac{2}{5}\right) = \frac{6}{5} \\ C = -\frac{4}{10} = -\frac{2}{5} \end{cases}$$

hence

$$y^* = \frac{8}{3}e^{3x} + \frac{6}{5}cosx - \frac{2}{5}sinx.$$

Then the general solution of $y'' - 3y' = 8e^{3x} + 4\, sin\, x$ is

$$y = y_0 + y^* = C_1 + C_2e^{3x} + \frac{8}{3}e^{3x} + \frac{6}{5}cosx - \frac{2}{5}sinx.$$

Answer: $y = C_1 + C_2e^{3x} + \frac{8}{3}e^{3x} + \frac{6}{5}cosx - \frac{2}{5}sinx.$