Answer on Question #58663 – Math – Differential Equations

Question

Find the general solution of the following differential equation

$$y'' - 4y' + 3y = 2(1+2x)e^x + x$$

Solution

The auxiliary equation is

$$\lambda^2 - 4\lambda + 3 = 0.$$

Its solutions are

$$\lambda_1 = \frac{4-\sqrt{16-12}}{2} = 1, \quad \lambda_2 = \frac{4+2}{2} = 3.$$

The general solution of the homogeneous differential equation $y'' - 4y' + 3y = 0$ is

$$Y = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}$$

$$Y = c_1 e^x + c_2 e^{3x},$$

where c_1, c_2 are arbitrary real constants.

The general solution of the non-homogeneous differential equation

$$y'' - 4y' + 3y = 2(1+2x)e^x + x$$

is

$$y = Y + \tilde{y},$$

Y is the general solution of the homogeneous differential equation;

$\tilde{y}$ is a particular solution of the non-homogeneous differential equation.

We use the method of undetermined coefficients.

Because $e^x = e^{x\lambda_1}$, we search a particular solution in the following form:

$\tilde{y} = (Ax^2 + Bx)e^x + Cx + D = Ax^2e^x + Bxe^x + Cx + D$

$\widetilde{y'} = (2Ax + B)e^x + (Ax^2 + Bx)e^x + C = Ax^2e^x + (2A + B)xe^x + Be^x + C$

$$\widetilde{y''} = 2Axe^x + Ax^2e^x + (2A+B)e^x + (2A+B)xe^x + Be^x = Ax^2e^x + 2(A+B)e^x + (4A+B)xe^x$$

Plug $\hat{y}, \widetilde{y'}, \widetilde{y''}$ into the initial non-homogeneous differential equation:

$$Ax^2e^x + 2(A+B)e^x + (4A+B)xe^x - 4(Ax^2e^x + (2A+B)xe^x + Be^x + C) +$$

$$+3(Ax^2e^x + Bxe^x + Cx + D) = 2(1+2x)e^x + x$$

$$-4Axe^x + 2(A+B)e^x + 3Cx + 3D - 4C = 4xe^x + 2e^x + x$$

Equate like terms and get the following system of equations:

$$\begin{cases} -4A = 4 \\ 2(A-B) = 2 \\ 3C = 1 \\ 3D - 4C = 0 \end{cases} \Rightarrow \begin{cases} A = -1 \\ B = -2 \\ C = \dfrac{1}{3} \\ D = \dfrac{4}{9} \end{cases}$$

Then

$$\tilde{y} = -x^2e^x - 2xe^x + \tfrac{x}{3} + \tfrac{4}{9}.$$

Finally get

$$y = Y + \tilde{y} = c_1e^x + c_2e^{3x} - x^2e^x - 2xe^x + \frac{x}{3} + \frac{4}{9}.$$

Answer: $y = c_1e^x + c_2e^{3x} - x^2e^x - 2xe^x + \tfrac{x}{3} + \tfrac{4}{9}.$