Answer on Question #58516 – Math – Algebra

Question 7

Find the term that contains x^5 in the expansion of $(2x+y)^{20}$.

Solution:

Let us write some terms in the given binomial expansion, as it contains not too many terms and also since the coefficient of the variables x and y namely 2 and 1 are small in value.

Apply the binomial theorem:

$$(2x+y)^{20} = {}^{20}C_0(y)^{20} + {}^{20}C_1(y)^{19}\cdot(2x)^1 + {}^{20}C_2(y)^{18}\cdot(2x)^2 +$$
$$+{}^{20}C_3(y)^{17}\cdot(2x)^3 + {}^{20}C_4(y)^{16}\cdot(2x)^4 + {}^{20}C_5(y)^{15}\cdot(2x)^5 + \ldots + {}^{20}C_{20}(2x)^{20}.$$

We have the following term:

$${}^{20}C_5(y)^{15}\cdot(2x)^5 = \frac{20!}{5!(20-5)!}\cdot(y)^{15}\cdot(2x)^5 = \frac{20!}{5!(15)!}\cdot y^{15}\cdot 32\cdot x^5;$$

$${}^{20}C_5(y)^{15}\cdot(2x)^5 = \frac{20\cdot19\cdot18\cdot17\cdot16\cdot15!}{5\cdot4\cdot3\cdot2\cdot1\cdot15!}\cdot 32\cdot x^5\cdot y^{15} = 496128\,x^5y^{15}.$$

Answer: The term that contains x^5 is $496128\,x^5y^{15}$.

Question 8

Find the coefficient of x^8 in the expansion of $(x^2 + 1/x)^{10}$.

Solution

We need to find the binomial coefficient of x^8 in $(x^2 + 1/x)^{10}$.

1. First, we have to find the term in which x^8 occurs. To find a term is equivalent to finding r. So, we will now find r.

2. To find r, we will need the general term. General term is:

$$T_{r+1} = {}^nC_r\cdot x^{n-r}\cdot y^r$$

3. In x^8, since the power of x is 8, we should notice that in the general term with x^{n-r} ,we should set 8 to be equal to n – r.

4. First collect all powers of x in the given binomial expansion $(x^2+x^{-1})^{10}$ using the general term.

Now, let us apply all four steps discussed above to find the binomial coefficient of x^8 in

($x^2 + x^{-1})^{10}$.

First, write the general term

$$T_{r+1} = {}^nC_r \cdot x^{n-r} \cdot y^r$$

From comparison, we see that x^2 in the given form stands for x in the standard form, while x^{-1} in the given form stands for y in the standard form. In short, $x = x^2$ and $y = x^{-1}$ Therefore, in the given binomial expansion $(x^2 + x^{-1})^{10}$ the general term is:

$$T_{r+1} = {}^nC_r \cdot (x^2)^{n-r} \cdot (x^{-1})^r$$

Now,

$$T_{r+1} = {}^nC_r \cdot (x^2)^{n-r} \cdot (x^{-1})^r = {}^nC_r \cdot (x^{2n-2r}) \cdot (x^{-1})^r =$$
$$= {}^nC_r \cdot (x^{2n-2r-r}) = {}^nC_r \cdot (x^{2n-3r}).$$

Since we need x^8, we observe that we must set $2n - 3r=8$.

Substitute n = 10,

$$\begin{aligned} &2n - 3r = 8; \\ &2 \cdot 10 - 3r = 8; \\ &20 - 8 = 3r; \\ &r = 4. \end{aligned}$$

Since r = 4, $T_{r+1} = T_{4+1} = T_5$.

So, it is the fifth term that contains x^8

Apply r = 4 in above, to write the binomial coefficient of x^8. It is

$$^{10}C_4 = \frac{10!}{4!(10-4)!} = \frac{10!}{4!(6)!} = \frac{10 \times 9 \times 8 \times 7 \times 6!}{4! \times 6!};$$

$$^{10}C_4 = \frac{10 \times 9 \times 8 \times 7}{4 \times 3 \times 2 \times 1} = 210.$$

Answer: The coefficient of x^8 is 210.

Question 9

Find the number of different ways of placing 15 balls in a row given that 4 are red, 3 are yellow, 6 are black and 2 are blue.

Solution

$$\left.\begin{array}{l}\text{red balls - 4}\\ \text{yellow balls - 3}\\ \text{black balls - 6}\\ \text{blue balls - 2}\end{array}\right\}\text{all 15 balls;}$$

To find the number of different ways of placing balls use the following formula:

$$C_n^k = \frac{n!}{k!(n-k)!}$$

Let us find the number of options to place 6 black balls:

$$C_{15}^{6} = \frac{15!}{6!(15-6)!} = \frac{15!}{6!(9)!}.$$

Let us find the number of options to fill the remaining seats red balls:

$$C_{15-6}^{4} = C_9^4 = \frac{9!}{4!(9-4)!} = \frac{9!}{4!(5)!}.$$

Let us find the number of options to fill the remaining space with yellow balls:

$$C_{9-4}^{3} = C_5^3 = \frac{5!}{3!(5-3)!} = \frac{5!}{3!(2)!}.$$

Blue balls is uniquely placed

Find the number of different ways of placing 15 balls in a row:

$$C_{15}^{6}\cdot C_{9}^{4}\cdot C_{5}^{3}=\frac{15!}{6!(9)!}\times\frac{9!}{4!(5)!}\times\frac{5!}{3!(2)!}=\frac{15!}{6!(4)!3!(2)!};$$

$$\frac{15!}{6!(4)!3!(2)!}=\frac{15\cdot14\cdot13\cdot12\cdot11\cdot10\cdot9\cdot8\cdot7\cdot6\cdot5\cdot4\cdot3\cdot2\cdot1}{6\cdot5\cdot4\cdot3\cdot2\cdot1\cdot4\cdot3\cdot2\cdot1\cdot3\cdot2\cdot1\cdot2\cdot1}=6306300;$$

$$C_{15}^{6}\cdot C_{9}^{4}\cdot C_{5}^{3}=6306300.$$

Answer: There are 6306300 different ways of placing 15 balls in a row.

Question 10

A pizza parlor offers the basic cheese pizza and a choice of 16 toppings. How many different kinds of pizza can be ordered at this pizza parlor?

Solution

The first method

We can either take every stuffing or not. There are two options (**N**) (yes and no).

n- types toppings (16).

We find the number of possible options for pizza:

$$\mathrm{N}^{n}=2^{16}=65536$$

The second method

- 0) no stuffing - C_{n}^{0};
- 1) one type of toppings - C_{n}^{1};
- 2) two types of toppings - C_{n}^{2};
- -------
- --------
- 16) all types toppings - C_{n}^{16}.

Then,

$$C_n^0 + C_n^1 + C_n^2 + .. + C_n^n = 2^n;$$
$$C_{16}^0 + C_{16}^1 + C_{16}^2 + .. + C_{16}^{16} = 2^{16} = 65536.$$

Answer: There are 65,536 different kinds of pizza at this pizza parlor.

Question 11

A committee of seven - consisting of a chairman, a vice chairman, a secretary and four other members - is to be chosen from a class of 20 students. In how many ways can this committee be chosen?

Solution

Placement of the n by m elements such places is called sampling, which by having m elements selected from among n data elements differ from one another, or composition elements or their arrangement order.

The number of placements of n by m A_n^m designated and defined by the formula:

$$A_n^m = n \cdot (n-1) \cdot (n-2) \cdot ... \cdot (n-m+1) = \frac{n!}{(n-m)!};$$

In our case, n=20 and m=7.

Then,

$$A_{20}^7 = 20 \cdot 19 \cdot 18 \cdot ... \cdot (20-7+1) = \frac{20!}{(20-7)!} = 390700800.$$

Answer: This committee can be chosen in 390700800 ways.

Question 12

What is the probability that in a class of 35 students, at least two have the same birthday?

Solution

Assume that the year is not a leap year and has 365 days.

If every birthday is unique, then with 35 students, there are 35 different days on which a birthday occurs.

P(at least two students with the same birthday) = 1 – P(all birthdays are unique)

The number of options when all have a unique birthday is:

$$365! = 365 \cdot 364 \cdot \ldots \cdot 1.$$

Any other option suits us. Number of options: 365^{35} .

P(all birthdays are unique) = P(unique birthday for 1st person selected) *P(unique birthday for 2nd person selected) *P(unique birthday for 3rd person selected) ... *P(unique birthday for 84th person selected) *P(unique birthday for 35th person selected)

$$\text{P(all birthdays are unique)} = \frac{365}{365} \times \frac{364}{365} \times \frac{365}{365} \times \ldots \times \frac{331}{365}.$$

From 365 to 331, there are (365-331) +1 = 35 numbers (need to add 1 since it is 365 to 281, inclusive)

In the denominator 365 will be multiplied together 35 times:

Denominator = 365^{35}

In the numerator:

$$\frac{365!}{330!} = \frac{365 \cdot 364 \cdot \ldots \cdot 330 \cdot 329 \cdot \ldots \cdot 2 \cdot 1}{330 \cdot 329 \cdot \ldots \cdot 2 \cdot 1} = \frac{365 \cdot 364 \cdot \ldots \cdot 331}{1}.$$

Combine the numerator and denominator:

$$\text{P}\left(\text{all birthdays are unique}\right) = \frac{\text{Numerator}}{Deno\min ator} = \frac{365!/330!}{365^{35}}.$$

Use the complement to find the answer to the original question:

P(at least two students with the same birthday) = 1 – P(all birthdays are unique)

$$\text{P}\left(\text{at least two students with the same birthday}\right) = 1 - \frac{365!/330!}{365^{35}}.$$

Answer: $\text{P}\left(\text{at least two students with the same birthday}\right) = 1 - \frac{365!/330!}{365^{35}}.$