Answer on Question #58435 – Math – Linear Algebra

Question

What is Gaussian elimination method and also Gauss-Jordan method?

Solution

The Gaussian elimination method

Consider a linear system.

1. Construct the augmented matrix for the system;

2. Use elementary row operations to transform the augmented matrix into a triangular one;

3. Write down the new linear system for which the triangular matrix is the associated augmented matrix;

4. Solve the new system. You may need to assign some parametric values to some unknowns, and then apply the method of back substitution to solve the new system.

Keep in mind that linear systems for which the matrix coefficient is upper-triangular are easy to solve. This is particularly true, if the matrix is in echelon form. So the trick is to perform elementary operations to transform the initial linear system into another one for which the coefficient matrix is in echelon form.

Example. Solve the following system by using the Gaussian elimination method.

$$\begin{cases} x+y+z &= 0 \\ x-2y+2z &= 4 \\ x+2y-z &= 2 \end{cases}$$

Consider the augmented matrix

$$\left(\begin{array}{rrr|r} 1 & 1 & 1 & 0 \\ 1 & -2 & 2 & 4 \\ 1 & 2 & -1 & 2 \end{array}\right).$$

Let us perform some elementary row operations on this matrix. Indeed, if we keep the first and second row, and subtract the first one from the last one we get

$$\left(\begin{array}{rrr|r} 1 & 1 & 1 & 0 \\ 1 & -2 & 2 & 4 \\ 0 & 1 & -2 & 2 \end{array}\right).$$

Next we keep the first and the last rows, and we subtract the first from the second. We get

$$\left(\begin{array}{rrr|r} 1 & 1 & 1 & 0 \\ 0 & -3 & 1 & 4 \\ 0 & 1 & -2 & 2 \end{array}\right).$$

Then we keep the first and second row, and we add the second to the third after multiplying it by 3 to get

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & -3 & 1 & 4 \\ 0 & 0 & -5 & 10 \end{array}\right).$$

This is a triangular matrix which is not in echelon form. The linear system for which this matrix is an augmented one is

$$\begin{cases} x+ & y+ & z & = & 0 \\ - & 3y+ & z & = & 4 \\ & - & 5z & = & 10 \end{cases}$$

This obviously implies z = -2. Substitute z = -2 into the second equation and from the second equation we get y = -2. Substitute z = -2 and y = -2 into the first equation and finally from the first equation we get

x = 4.

Therefore the linear system has one solution

$$x = 4, \; y = -2, \; z = -2.$$

Going from the last equation to the first while solving for the unknowns is called **backsolving**.

The Gauss-Jordan elimination method

The Gauss-Jordan elimination method to solve a system of linear equations is described in the following steps.

1. Write the augmented matrix of the system.

2. Use row operations to transform the augmented matrix in the form described below, which is called the reduced row echelon form (RREF).

(a) The rows (if any) consisting entirely of zeros are grouped together at the bottom of the matrix.

(b) In each row that does not consist entirely of zeros, the leftmost nonzero element is a 1 (called a leading 1 or a pivot).

(c) Each column that contains a leading 1 has zeros in all other entries.

(d) The leading 1 in any row is to the left of any leading 1's in the rows below it.

3. Stop process in step 2 if you obtain a row whose elements are all zeros except the last one on the right. In that case, the system is inconsistent and has no solutions. Otherwise, finish step 2 and read the solutions of the system from the final matrix.

Example. Solve the following system by using the Gauss-Jordan elimination method.

$$\begin{cases} x + y + z = 5 \\ 2x + 3y + 5z = 8 \\ 4x + 5z = 2 \end{cases}$$

Solution: The augmented matrix of the system is the following.

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 5 \\ 2 & 3 & 5 & 8 \\ 4 & 0 & 5 & 2 \end{array}\right]$$

We will now perform row operations until we obtain a matrix in reduced row echelon form.

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 5 \\ 2 & 3 & 5 & 8 \\ 4 & 0 & 5 & 2 \end{array}\right] \xrightarrow{R_2 - 2R_1} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 5 \\ 0 & 1 & 3 & -2 \\ 4 & 0 & 5 & 2 \end{array}\right]$$

$$\xrightarrow{R_3 - 4R_1} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 5 \\ 0 & 1 & 3 & -2 \\ 0 & -4 & 1 & -18 \end{array}\right]$$

$$\xrightarrow{R_3 + 4R_2} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 5 \\ 0 & 1 & 3 & -2 \\ 0 & 0 & 13 & -26 \end{array}\right]$$

$$\xrightarrow{\frac{1}{13}R_3} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 5 \\ 0 & 1 & 3 & -2 \\ 0 & 0 & 1 & -2 \end{array}\right]$$

$$\xrightarrow{R_2 - 3R_3} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 5 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & -2 \end{array}\right]$$

$$\xrightarrow{R_1 - R_3} \left[\begin{array}{ccc|c} 1 & 1 & 0 & 7 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & -2 \end{array}\right]$$

$$\xrightarrow{R_1 - R_2} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & -2 \end{array}\right]$$

From this final matrix, we can read the solution of the system. It is

$$\boxed{x = 3, \quad y = 4, \quad z = -2.}$$