Answer on Question #58207 – Math – Differential Equations

Question

Determine whether the following PDE can be reduced to a set of two ODEs by the method of separation of variables.

i) d^2 u / dx^2 + d^2 u / dy^2 = x

Solution

i) Let

$$u(x, y) = X(x)Y(y).$$

Then,

$$\frac{\partial^2 u}{\partial x^2} = Y\frac{d^2 X}{dx^2} = YX''; \qquad \frac{\partial^2 u}{\partial y^2} = X\frac{d^2 Y}{dy^2} = X\ddot{Y}.$$

Now,

$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = x$

can be rewritten in the following form:

$$Y\frac{d^2 X}{dx^2} + X\frac{d^2 Y}{dy^2} = x$$

$$\frac{Y\frac{d^2 X}{dx^2}}{XY} + \frac{X\frac{d^2 Y}{dy^2}}{XY} = \frac{x}{XY}$$

$$\frac{\frac{d^2 X}{dx^2}}{X} + \frac{\frac{d^2 Y}{dy^2}}{Y} = \frac{x}{XY}$$

$$\frac{X''}{X} + \frac{\ddot{Y}}{Y} = \frac{x}{XY}$$

We cannot reduce $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = x$ to a set of two ODEs by the method of separation of variables.

Answer: No.

Question

Determine whether the following PDE can be reduced to a set of two ODEs by the method of separation of variables.

ii) x d^2 u / dx^2 + t du/dt = 0

Solution

ii) Let $u(x, t) = X(x)T(t)$. Then,

$$\frac{\partial^2 u}{\partial x^2} = T\frac{d^2 X}{dx^2} = X'' \cdot T; \quad \frac{\partial u}{\partial t} = X\frac{dT}{dt} = X \cdot \dot{T}.$$

Now,

$x\frac{\partial^2 u}{\partial x^2} + t\frac{\partial u}{\partial t} = 0$

can be rewritten in the following form:

$$xT\frac{d^2 X}{dx^2} + tX\frac{dT}{dt} = 0,$$

$$\frac{xT\frac{d^2 X}{dx^2}}{XT} + \frac{tX\frac{dT}{dt}}{XT} = 0,$$

$$\frac{x\frac{d^2 X}{dx^2}}{X} + \frac{t\frac{dT}{dt}}{T} = 0,$$

$$\frac{x\frac{d^2 X}{dx^2}}{X} = -\frac{t\frac{dT}{dt}}{T},$$

$$\frac{xX''}{X} = -\frac{t\dot{T}}{T}.$$

The left-hand side is the function of x and the right-hand side is the function of t, therefore, now both sides must be constant, so we set

$$\frac{xX''}{X} = -\frac{t\dot{T}}{T} = -\lambda.$$

From these we get the ordinary differential equations:

$$\frac{xX''}{X} = -\lambda,$$

$$\frac{t\dot{T}}{T} = \lambda,$$

that is,

$$xX'' + \lambda X = 0,$$

$$t\dot{T} - \lambda T = 0.$$

We can reduce $x\frac{\partial^2 u}{\partial x^2} + t\frac{\partial u}{\partial t} = 0$ to a set of two ODEs by the method of separation of variables.

Answer: Yes.