Answer on Question #57913 - Math - Abstract Algebra

Assume that A and B are isomorphic commutative rings with unity. Prove that if A is a field, so is B.
Solution.
By definition field is a
1. commutative ring with identity.
2. $\forall\, a \in A, a \neq 0, \exists\, a^{-1}: a\, a^{-1} = 1$

By condition A and B isomorphic, i. e exists $f: A \to B$ a

$$f(a)f(b) = f(ab)$$

$$f(a) + f(b) = f(a + b)$$

, where $a, b \in A$

Evidently that $f(0_A) = 0_B, f(1_A) = 1_B$ and $\forall\, b \in B,\ \exists\, a \in A : f(a) = b$

So we only need to prove that for each $b \in B, b \neq 0$ exists $b^{-1} \in B: b\, b^{-1} = 1$

Lets take for each $b \in B, b \neq 0$ the $b' = f(a^{-1}),$ where $f(a) = b$.
It is possible because $b \neq 0$, than $a \neq 0$, and a^{-1} exists.

$b\, b' = f(a)f(a^{-1}) = f(a\, a^{-1}) = f(1) = 1$, i.e. we found b^{-1}

Answer: We proved that for each nonzero element in B exists inverse element. And it means that B is a field too.