Answer on Question #57663 – Math – Calculus

Find the Fourier series of $f(x) = 2 - x^2$ on $(-2 < x < 2)$

Solution

Recall that the mathematical expression

$$f(x) \sim a_0 + \sum_{i=1}^{\infty} \left(a_i \cos\left(\frac{\pi}{2} ix\right) + b_i \sin\left(\frac{\pi}{2} ix\right) \right)$$

is called a **Fourier series**. The constants a_0, a_i and b_i for $i = 1,2,3 \dots.$, are called the **coefficients** of $F_n(x)$.

So we need to find this coefficients.

$$a_0 = \frac{1}{2} \int_{-2}^{2} f(x) dx = \frac{1}{2} \int_{-2}^{2} (2 - x^2) dx = \frac{1}{2} \left(2x - \frac{x^3}{3} \right) \Bigg|_{-2}^{2} = \frac{4}{3}$$

$$a_i = \frac{1}{2} \int_{-2}^{2} f(x) cos\left(\frac{\pi}{2} ix\right) dx$$

$$= \frac{1}{2} \int_{-2}^{2} (2 - x^2) cos\left(\frac{\pi}{2} ix\right) dx = \frac{1}{2} \int_{-2}^{2} 2cos\left(\frac{\pi}{2} ix\right) dx$$

$$= -\frac{1}{2} \int_{-2}^{2} x^2 cos\left(\frac{\pi}{2} ix\right) dx = 0 - \frac{1}{2} \int_{-2}^{2} x^2 cos\left(\frac{\pi}{2} ix\right) dx$$

$$= \frac{2}{2\pi i} \int_{-2}^{2} x^2 d\left(sin\left(\frac{\pi}{2} ix\right) \right)$$

$$= \frac{1}{\pi i} \left(x^2 sin\left(\frac{\pi}{2} ix\right) \Bigg|_{-2}^{2} - \int_{-2}^{2} 2x sin\left(\frac{\pi}{2} ix\right) d(x) \right)$$

$$= \frac{4}{(\pi i)^2} \int_{-2}^{2} x\, d\left(cos\left(\frac{\pi}{2} ix\right)\right) = \frac{4}{(\pi i)^2}\left(xcos\left(\frac{\pi}{2} ix\right)\Big|_{-2}^{2} - \int_{-2}^{2} cos\left(\frac{\pi}{2} ix\right)\, d(x)\right)$$

$$= \frac{4}{(\pi i)^2} 4cos(\pi i) = (-1)^i \frac{16}{(\pi i)^2}$$

$$b_i = \frac{1}{2} \int_{-2}^{2} f(x) sin\left(\frac{\pi}{2} ix\right) dx$$

$$= \frac{1}{2} \int_{-2}^{2} (2 - x^2) sin\left(\frac{\pi}{2} ix\right) dx$$

$$= \frac{1}{2} \int_{-2}^{2} 2sin\left(\frac{\pi}{2} ix\right) dx - \frac{1}{2} \int_{-2}^{2} x^2 sin\left(\frac{\pi}{2} ix\right) dx = 0$$

The coefficients b_i are zero, because both functions are odd in symmetric integrating limits.

Hence

$$f(x) = 2 - x^2 \sim \frac{4}{3} + \sum_{i=1}^{\infty}\left((-1)^i \frac{16}{(\pi i)^2} \cos\left(\frac{\pi}{2} ix\right) + 0 * \sin\left(\frac{\pi}{2} ix\right)\right)$$

$$= \frac{4}{3} + \sum_{i=1}^{\infty}\left((-1)^i \frac{16}{(\pi i)^2} \cos\left(\frac{\pi}{2} ix\right)\right)$$

$$= \frac{4}{3} - \frac{16}{\pi^2} \cos\left(\frac{\pi}{2} x\right) + \frac{16}{4\pi^2} \cos(\pi x) - \frac{16}{9\pi^2} \cos\left(\frac{3\pi}{2} x\right) + \cdots$$

Answer: $f(x) \sim \frac{4}{3} - \frac{16}{\pi^2} \cos\left(\frac{\pi}{2} x\right) + \frac{16}{4\pi^2} \cos(\pi x) - \frac{16}{9\pi^2} \cos\left(\frac{3\pi}{2} x\right) + \cdots$