Answer on Question #57662 – Math – Calculus

Find the Fourier series of $f(x)=|x|$ on $[-\pi;\pi]$.

Solution

$$f(x)=|x|=\begin{cases} x, & x\ge 0 \\ -x, & x<0 \end{cases}.$$

Let's expand the function $f(x)=|x|$ in the Fourier series with the period $T=2l$, where l is a positive arbitrary number.

$f(x)=|x|$ is the even function, and we can expand it as $f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos\frac{\pi nx}{l}$,

where $a_0=\frac{2}{l}\int_0^l f(x)dx, a_n=\frac{2}{l}\int_0^l f(x)\cos\frac{\pi nx}{l}dx.$

Note that integration is over the positive interval, where $f(x)=x$.

$$a_0=\frac{2}{l}\int_0^l xdx=\frac{2}{l}\frac{x^2}{2}\Big|_0^l=l,$$

$$\int_a^b udv=uv\Big|_a^b-\int_a^b vdu$$

$$a_n=\frac{2}{l}\int_0^l x\cos\frac{\pi nx}{l}dx=\frac{2}{l}\int_0^l \underbrace{x}_{u}\underbrace{\cos\frac{\pi nx}{l}dx}_{dv}=\left|\begin{matrix} u=x & dv=\cos\frac{\pi nx}{l}dx \\ du=dx & v=\frac{l}{\pi n}\sin\frac{\pi nx}{l}\end{matrix}\right|=$$

$$=\frac{2}{l}\frac{l}{\pi n}x\sin\frac{\pi nx}{l}\Big|_0^l-\frac{2}{l}\frac{l}{\pi n}\int_0^l\sin\frac{\pi nx}{l}dx=\frac{2}{\pi n}x\sin\frac{\pi nx}{l}\Big|_0^l-\frac{2}{\pi n}\int_0^l\sin\frac{\pi nx}{l}dx=$$

$$=\frac{2}{\pi n}(l\underbrace{\sin\pi n}_{0}-0)+\frac{2l}{(\pi n)^2}\cos\frac{\pi nx}{l}\Big|_0^l=\frac{2l}{(\pi n)^2}(\cos\pi n-1)=\frac{2l}{(\pi n)^2}((-1)^n-1).$$

Note that $\sin\pi n=0,\ \cos\pi n=(-1)^n$ if n is an integer.

We can rewrite a_n as

$$a_n=\frac{2l}{(\pi n)^2}((-1)^n-1)=\begin{cases} 0, & \text{if } n \text{ is even} \\ -\frac{4l}{(\pi n)^2}, & \text{if } n \text{ is odd}\end{cases}=\begin{cases} 0, & \text{if } n=2k \\ -\frac{4l}{\pi^2(2k-1)^2}, & \text{if } n=2k-1\end{cases}$$

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos\frac{\pi nx}{l}=\frac{l}{2}+\sum_{n=1}^{\infty}\left(\frac{2l}{(\pi n)^2}((-1)^n-1)\right)\cos\frac{\pi nx}{l}=$$

$$=\frac{l}{2}+\frac{2l}{\pi^2}\sum_{n=1}^{\infty}\frac{((-1)^n-1)}{(n)^2}\cos\frac{\pi nx}{l}=\frac{l}{2}+\frac{4l}{\pi^2}\sum_{k=1}^{\infty}\frac{1}{(2k-1)^2}\cos\frac{\pi(2k-1)x}{l}.$$

If $l=\pi$, then

$$f(x)=\frac{\pi}{2}+\frac{4\pi}{\pi^2}\sum_{k=1}^{\infty}\frac{1}{(2k-1)^2}\cos\frac{\pi(2k-1)x}{\pi}=$$

$$=\frac{\pi}{2}+\frac{4}{\pi}\sum_{k=1}^{\infty}\frac{1}{(2k-1)^2}\cos(2k-1)x.$$

Answer: $f(x)=\frac{\pi}{2}+\frac{4}{\pi}\sum_{k=1}^{\infty}\frac{1}{(2k-1)^2}\cos(2k-1)x.$