Answer on Question #57661 – Math – Calculus

Q.1

Find the Fourier series of $f(x) = 2 - \boldsymbol{x^2}$ on $(-2 < x < 2)$

Solution

We have

$$F(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty}(a_k \cos\frac{k\pi x}{l} + b_k \sin\frac{k\pi x}{l})$$

where

$$a_k = \frac{1}{l}\int_{-l}^{l} f(x)\cos\frac{k\pi x}{l}\,dx,$$

$$b_k = \frac{1}{l}\int_{-l}^{l} f(x)\sin\frac{k\pi x}{l}\,dx\ .$$

In our case

$$l = 2.$$

Because

$$f(-x) = 2 - (-x)^2 = \mathbf{2} - x^2 = f(x)$$

then

$$b_k = 0,$$

$$a_k = \int_0^2 f(x)\cos\frac{k\pi x}{2}\,dx.$$

Thus

$$a_0 = \int_0^2 f(x)\,dx = \int_0^2 (\mathbf{2} - x^2)dx = (2x - \frac{x^3}{3})\Big|_0^2 = 4 - \frac{8}{3} = \frac{4}{3};$$

$$a_k = \int_0^2 (\mathbf{2} - x^2)\cos\frac{k\pi x}{2}\,dx = \frac{2}{k\pi}\int_0^2 (\mathbf{2} - x^2)\,d(\sin\frac{k\pi x}{2}) =$$

$$= \frac{2}{k\pi}(\mathbf{2} - x^2)\sin\frac{k\pi x}{2}\Big|_0^2 + \frac{4}{k\pi}\int_0^2 x\sin\frac{k\pi x}{2}\,dx = -\frac{8}{k^2\pi^2}\int_0^2 x\,d(\cos\frac{k\pi x}{2}) =$$

$$= -\frac{8}{k^2\pi^2}x\cos\frac{k\pi x}{2}\Big|_0^2 + \frac{8}{k^2\pi^2}\int_0^2 \cos\frac{k\pi x}{2}\,dx = -\frac{16}{k^2\pi^2}\cos k\pi = \frac{16}{k^2\pi^2}(-1)^{k+1}.$$

And finally

$$F(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos\frac{k\pi x}{2} = \frac{2}{3} + \frac{16}{\pi^2}\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k^2}\cos\frac{k\pi x}{2}.$$

Answer:

$$\frac{2}{3} + \frac{16}{\pi^2}\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k^2}\cos\frac{k\pi x}{2}$$

Q.2

Find the Fourier series of $f(x) = |x|$ on $[-\pi; \pi]$

Solution:

We have

$$F(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty}(a_k \cos kx + b_k \sin kx)$$

where

$$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \cos kx \, dx,$$

$$b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \sin kx \, dx \, .$$

Because

$$f(-x) = |-x| = |x| = f(x)$$

then

$$b_k = 0,$$

$$a_k = \frac{2}{\pi}\int_{0}^{\pi} f(x) \cos kx \, dx.$$

Thus

$$a_0 = \frac{2}{\pi}\int_{0}^{\pi} f(x) \, dx = \frac{2}{\pi}\int_{0}^{\pi} x \, dx = \frac{2}{\pi} \cdot \frac{x^2}{2}\bigg|_0^{\pi} = \pi;$$

$$a_k = \frac{2}{\pi}\int_{0}^{\pi} f(x) \cos kx \, dx = \frac{2}{\pi}\int_{0}^{\pi} x \cos kx \, dx = \frac{2}{k\pi}\int_{0}^{\pi} x \, d(\sin kx) =$$

$$= \frac{2}{k\pi} x \sin kx|_0^{\pi} + \frac{2}{k\pi}\int_{0}^{\pi} \sin kx \, dx =$$

$$= -\frac{2}{k^2\pi} \cos kx|_0^{\pi} = -\frac{2}{k^2\pi} \cdot (\cos k\pi - 1) = -\frac{2}{k^2\pi} \cdot ((-1)^k - 1).$$

So

$$a_k = \begin{cases} \dfrac{4}{(2n+1)^2\pi}, & k = 2n+1; \\ 0, & k = 2n. \end{cases}$$

And finally

$$F(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos kx = \frac{\pi}{2} + \frac{4}{\pi}\sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} \cos(2n+1)x.$$

Answer:

$$\frac{\pi}{2} + \frac{4}{\pi}\sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} \cos(2n+1)x$$