Answer on Question #57141 - Math - Analytic Geometry

Show that the line through the centre perpendicular to the normal at any point does not meet the hyperbola.

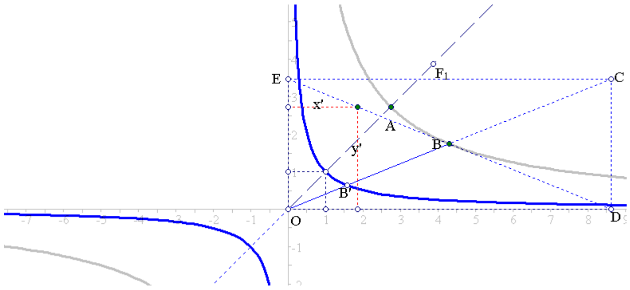
Solution

Suppose that there is a hyperbola of the form $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.

A_1 (x_1, y_1) is the point of hyperbola. The tangent line $\mathrm{d}_1: \frac{xx_1}{a^2} - \frac{yy_1}{b^2} = 1$.

The tangent line at the point A_1 doesn't have any other common points.

Prove "hyperbole and a tangent have only one common point".



Define the symmetric C of O w.r. to B(x_0,y_0) and build the rectangle ODCE. The diagonal DE is tangent to the hyperbola at B. In fact, at B it has a common point with the hyperbola. Then, an arbitrary point (x',y') on the line DE has y'/(2x_0-x') = y_0/x_0 = a^2/$x_0$2, where x_0*y_0 = a^2 is the equation of the hyperbola. Thus y' = (a^2(2x_0-x'))/x_0^2 < a^2/x' <==> 0 < (x_0-x')^2 , which means that y' is below the hyperbola point (x', a^2/x'). AD having only one common point (B) with the hyperbola is tangent to it.

The line through the centre perpendicular to the normal at any point is a: $\mathrm{y} = \mathrm{mx}$ because the centre is the origin. The line a is perpendicular normal line at the point A_1. Obviously the line a is parallel to the tangent line at the point A_1.

Let A_2 and A_1 are symmetric to each other with respect to the center.

A_2 (x_2, y_2) => A_2 (-x_1, -y_1). The tangent line $\mathrm{d}_2: \frac{xx_1}{a^2} - \frac{yy_1}{b^2} = -1$

d_1 and d_2 are parallel because their slopes are the same. Here we obtain the coefficients that differ only in sign. The tangent lines have only one common point with the hyperbola. The line a lies between two tangents d_1 and d_2. The line a does not meet the hyperbola.