Answer on Question #57136 – Math – Calculus

Question

If $f(x,y)=\ln\left(\frac{x^2+y^2}{xy}\right)$, then prove that $\frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2}=0$.

Solution

$f(x,y)=\ln\left(\frac{x^2+y^2}{xy}\right)=\ln\left(\frac{x}{y}+\frac{y}{x}\right)$. Compute

$$\frac{\partial f}{\partial x}=\frac{1}{\left(\frac{x^2+y^2}{xy}\right)}\cdot\left(\frac{1}{y}-\frac{y}{x^2}\right)=\frac{x}{x^2+y^2}-\frac{y^2}{x(x^2+y^2)}=\frac{x^2-y^2}{x(x^2+y^2)}=\frac{x^2+y^2-2y^2}{x(x^2+y^2)}=\frac{1}{x}-2\frac{y^2}{x(x^2+y^2)}$$

and

$$\frac{\partial f}{\partial y}=\frac{1}{\left(\frac{x^2+y^2}{xy}\right)}\cdot\left(-\frac{x}{y^2}+\frac{1}{x}\right)=-\frac{x^2}{y(x^2+y^2)}+\frac{y}{x^2+y^2}=\frac{y^2-x^2}{y(x^2+y^2)}=\frac{x^2+y^2-2x^2}{y(x^2+y^2)}=\frac{1}{y}-2\frac{x^2}{y(x^2+y^2)}.$$

Thus,

$$\frac{\partial^2 f}{\partial x^2}=\frac{\partial}{\partial x}\left(\frac{1}{x}-2\frac{y^2}{x(x^2+y^2)}\right)=\frac{\partial}{\partial x}\left(\frac{1}{x}-2\frac{y^2}{(x^3+xy^2)}\right)=-\frac{1}{x^2}+\frac{2y^2}{(x^3+xy^2)^2}\cdot(3x^2+y^2)=-\frac{1}{x^2}+\frac{2y^2}{x^2(x^2+y^2)^2}\cdot(3x^2+y^2)=\frac{-(x^4+2x^2y^2+y^4)+6x^2y^2+2y^4}{x^2(x^2+y^2)^2}=\frac{y^4+4x^2y^2-x^4}{x^2(x^2+y^2)^2}$$

and

$$\frac{\partial^2 f}{\partial y^2}=\frac{x^4+4x^2y^2-y^4}{y^2(x^2+y^2)^2}.$$

Then

$$\frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2}=\frac{y^4+4x^2y^2-x^4}{x^2(x^2+y^2)^2}+\frac{x^4+4x^2y^2-y^4}{y^2(x^2+y^2)^2}=\frac{y^6+4x^2y^4-x^4y^2+x^6+4x^4y^2-y^4x^2}{x^2y^2(x^2+y^2)^2}=\frac{y^6+3x^2y^4+x^6+3x^4y^2}{x^2y^2(x^2+y^2)^2}=$$

$\frac{(x^2+y^2)^3}{x^2y^2(x^2+y^2)^2}=\frac{1}{x^2}+\frac{1}{y^2}\neq 0$, hence the statement of the question is false.