Answer on Question #57117 – Math – Calculus

Question

Let

$$f(x) = \begin{cases} 1 + \frac{2x}{\pi} & -\pi < x < 0 \\ 1 - \frac{2x}{\pi} & 0 < x < \pi \end{cases} \qquad \textbf{(1)}$$

a) Show that f is even

b) Find the Fourier series for f.

Solution

a) A function f is even if and only if $f(-\mathrm{x}) = f(\mathrm{x})$.

The function (1) can be rewritten in the form

$$f(x) = 1 - \frac{2}{\pi}|x|, \qquad x \in (-\pi, \pi)\backslash\{0\}$$

Since $|-\mathrm{x}| = |\mathrm{x}|$, we have $f(-\mathrm{x}) = 1 - \frac{2}{\pi}|-\mathrm{x}| = 1 - \frac{2}{\pi}|\mathrm{x}| = f(\mathrm{x})$ and the function $f(\mathrm{x})$ is even.

b) Because the function $f(\mathrm{x})$ in (1) is even, its Fourier series has form

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx,$$

where $a_0 = \frac{2}{\pi}\int_0^\pi f(x)dx$, $a_n = \frac{2}{\pi}\int_0^\pi f(x)\cos nx\, dx$, $n \geq 1$.

Calculate coefficients

$$a_0 = \frac{2}{\pi}\int_0^\pi f(x)dx = \frac{2}{\pi}\int_0^\pi \left(1 - \frac{2x}{\pi}\right)dx = \frac{2x}{\pi}\bigg|_0^\pi - \frac{2}{\pi}\cdot\frac{x^2}{\pi}\bigg|_0^\pi = \frac{2\pi}{\pi} - \frac{2}{\pi}\cdot\frac{\pi^2}{\pi} = 2 - 2 = 0.$$

and

$$a_n = \frac{2}{\pi}\int_0^\pi f(x)\cos nx\, dx = \frac{2}{\pi}\int_0^\pi \left(1 - \frac{2x}{\pi}\right)\cos nx\, dx = \frac{2}{\pi}\int_0^\pi \cos nx\, dx - \frac{4}{\pi^2}\int_0^\pi x\cos nx\, dx. \qquad \textbf{(2)}$$

The first integral in (2) equals zero:

$$\int_0^\pi \cos nx\, dx = \frac{1}{n}\sin nx\bigg|_0^\pi = \frac{1}{n}(\sin(n\cdot\pi) - \sin(n\cdot 0)) = \frac{1}{n}(0-0) = 0, n \geq 1.$$

The second integral in (2) is calculated using integration by parts with $u = x$, $dv = \cos nx\, dx$ and $du = dx$, $v = \frac{1}{n}\sin nx$:

$$\int_0^\pi x\cos nx\,dx = \frac{x}{n}\sin nx\Big|_0^\pi - \frac{1}{n}\int_0^\pi \sin nx\,dx = \frac{\pi\sin(n\pi)}{n} - \frac{0\cdot\sin(n\cdot 0)}{n} + \frac{1}{n^2}\cos nx\Big|_0^\pi$$
$$= 0 + \frac{1}{n^2}(\cos n\pi - \cos 0) = \frac{(-1)^n - 1}{n^2}$$

Substitute for these values into expression a_n in (2) and obtain

$$a_n = -\frac{4}{\pi^2}\cdot\frac{(-1)^n - 1}{n^2} = \frac{4(1-(-1)^n)}{\pi^2 n^2}$$

Because $(-1)^n = 1$ for every even n, Fourier series for f contains only odd terms:

$$a_n = \begin{cases} \frac{8}{\pi^2 n^2}, & n = 2k-1 \\ 0, & n = 2k \end{cases}$$

Thus, Fourier series for f is

$$f(x) = \frac{8}{\pi^2}\sum_{k=1}^{\infty}\frac{\cos(2k-1)x}{(2k-1)^2}$$